University of Illinois **Spring 2024**

ECE 313: Problem Set 1

Due: Friday, January 26 at 07:00:00 p.m.
Reading: *ECE 313 Course Notes,* Chapter 1

Note on reading: For most sections of the course notes there are short answer questions at the end of the chapter. We recommend that after reading each section you try answering the short answer questions. Do not hand these in; answers to the short answer questions are provided in the appendix of the notes.

Note on turning in homework: Homework is assigned on a weekly basis on Fridays, and is due by 7 p.m. on the following Friday. You must upload handwritten homework to Gradescope. Alternatively, you can typeset the homework in LaTeX. However, no additional credit will be awarded to typeset submissions. No late homework will be accepted.
Please write on the top right corner of the first page:
NAME
NETID
SECTION
PROBLEM SET #
Page numbers are encouraged but not required. Five points will be deducted for improper headings. Please assign your uploaded pages to their respective question numbers while submitting your homework on Gradescope. **5 points will be deducted for incorrectly assigned page numbers.**

1. **[Basics of Calculus]**
   Solve the following problems:
   (a) Prove that $1 + x + x^2 + \cdots + x^{n-1} = \frac{1-x^n}{1-x}$ for all $x \neq 1$ and all integers $n \geq 1$.
   (b) Continuing to assume that n is a positive integer, what is the value of the sum $1 + x + x^2 + \cdots + x^{n-1}$ when $x = 1$? Does your answer equal $\lim_{x \to 1} \frac{1-x^n}{1-x}$? (Hint: use L'Hôpital's rule)
   (c) Assuming that $|x| < 1$, find the sum of the series $1 + x + x^2 + \cdots$. (Hint: it is the limit of the finite sum $1 + x + x^2 + \cdots + x^{n-1}$ as $n \to \infty$).
   (d) Find the derivative of $\exp(-x) \sum_{k=0}^{n} \frac{x^k}{k!}$. Simplify your answer as much as possible. Additionally, find the derivative of $\exp(-x) \sum_{k=0}^{\infty} \frac{x^k}{k!}$.

2. **[Solving definite integrals]**
   Find the value of the following definite integrals. Be sure to show your work.
   (a) $\int_{-2}^{3} |x^2 - 1| dx$
   (b) $\int_{-1}^{0} x^2 (1 + x^3)^{10} dx$
   (c) $\int_{0}^{1} x \exp(-x) dx$
   (d) $\int_{-\pi}^{\pi} x^3 \sin(1 - x^4) dx$

3. **[Derivatives and integrals]**
   Let $\frac{d}{dx} f(x) = g(x)$, $-\infty < x < \infty$, and let C denote an arbitrary constant. Which of the following statements is true for all x? (Assume $f(x) > 0$ for (f).)

(a) $\frac{d}{dx}f(-x) = -g(-x)$

(b) $\frac{d}{dx}f(x^3/3) = x^2 g(x^3/3)$

(c) $\frac{d}{dx}\exp(f(x^2)) = g(x^2)\exp(f(x^2))$

(d) $\int g(-x)dx = f(-x) + C$

(e) $\int g(x^3/3)dx = \frac{f(x^3/3)}{x^2} + C$

(f) $\int \frac{g(x)}{f(x)}dx = \ln(f(x)) + C$

4. **[Defining a set of outcomes I]**
Suppose four teams, numbered 1 through 4, play a single-elimination tournament consisting of three games. Two teams play each game and one of them always wins i.e. ties do not occur. Teams 1 and 2 play each other in the first game; then, the winner of the first game plays against team 3 in the second game; finally, the winner of the second game plays against team 4 in the third game.

   (a) Define a sample space Ω so that the elements of Ω correspond to the possible scenarios of this tournament. An element of Ω should specify the entire sequence of games, not just the final winner. Give a brief explanation of how your notation lines up with the situation.

   (b) Determine $|\Omega|$, the cardinality of Ω (i.e. how many different scenarios there are).

5. **[Defining a set of outcomes II]**
A random experiment consists of selecting two balls in succession from an urn containing two blue balls and one red ball. Assume that each ball from the urn is equally likely to be chosen.

   (a) Suppose that the balls are not replaceable, i.e., the chosen ball in the first selection is removed from the urn. What is the sample space Ω for this experiment?

   (b) Suppose now that the balls are replaceable, i.e., the chosen ball in the first selection is immediately put back into the urn. What is the sample space Ω for this experiment?

   (c) Considering both of these experiments, does the outcome of the first draw affect the outcome of the second draw? Please briefly justify your answer for both cases.

6. **[Possible probability assignments I]**
Suppose A and B are events for some probability space such that $P(AB) = 0.5$ and $P(A \cup B) = 0.8$. Specify the domain of $P(A)$ (i.e. what values it can take), express $P(B)$ as a function of $P(A)$, and sketch that function. Hint: Use a Karnaugh Map.

7. **[Possible probability assignments II]**
A random experiment has a sample space $\Omega = \{a, b, c, d\}$. Suppose that $P(\{b, c, d\}) = \frac{5}{8}$, $P(\{a, b\}) = \frac{1}{2}$, and $P(\{b, c\}) = \frac{1}{4}$. Use the axioms of probability to find the probabilities of the elementary events ($P(\{a\})$, $P(\{b\})$, $P(\{c\})$, and $P(\{d\})$).

8. **[Displaying outcomes in a two event Karnaugh map]**
Two fair 4-sided dice are rolled. Let A be the event that both numbers rolled are strictly less than 3, and let B be the event that both numbers rolled are equal to each other.

   (a) Display the outcomes in a Karnaugh map.

   (b) Determine $P(AB^c)$ and $P(A^cB)$.