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Counting Problems & Fine-Grained Complexity

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SAT vs. #SAT

NP-complete — #P-complete

perfect matching vs. #perfect matching

in P

Core Problems in FGC:

1. 3SUM
   (given n numbers S, $?\exists a,b,c \in S$ s.t. $a+b+c=0$)
   Conj: $\Omega(n^{2-\varepsilon})$
2. APSP in weighted, dense graphs
   Conj: $\Omega(n^{3-\varepsilon})$
3. ZWT (Zero-Weight Triangle)
   Conj: $\Omega(n^{3-\varepsilon})$

$$\begin{tikzcd} & \text{ZWT} & \\ \text{3SUM} \arrow[ur] & & \text{APSP} \arrow[ul] \end{tikzcd}$$

Q: 3SUM vs. #3SUM
APSP vs. #APSP
ZWT vs. #ZWT
⋮

New Thm (i) 3SUM could be solved in $O(n^{2-\delta})$ time
$\Leftrightarrow$ #3SUM " " " $O(n^{2-\delta'})$ time

(ii) ZWT . . . . . . $O(n^{3-\delta})$.
$\Leftrightarrow$ #ZWT . . . . $O(n^{3-\delta'})$ time.

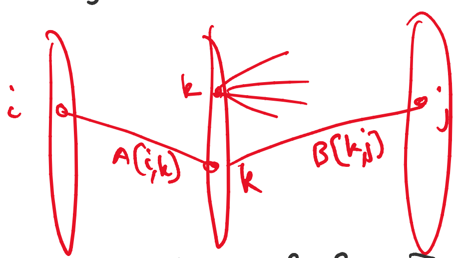
⋮

Lemma ("Equality Matrix Product") [Matoušek '91]

Given $n \times n$ matrices A, B,
can compute $C[i,j] = \left|\{k : A[i,k] = B[k,j]\}\right|$
in $O(n^{2.69})$ time.

Can compute $C[i,j] = |\{k : A(i,k) = B(k,j)\}|$
in $O(n^{2.69})$ time.

Pf: idea - "high-low trick"

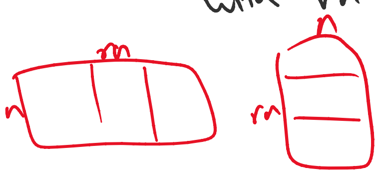


let F_k = all elems of freq $> n/r$
in $B[k,1], \dots, B[k,n]$

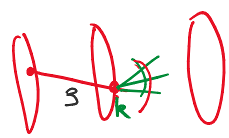
$|F_k| \le r$.

High Case: $C[i,j] = \sum_k \sum_{p \in F_k} \underbrace{[A[i,k]=p]}_{A'[i,(k,p)]} \cdot \underbrace{(B[k,j]=p]}_{B'[(k,p),j]}$

standard product of $n \times rn$
with $rn \times n$



$\Rightarrow O(r n^{\omega})$



Low case: for each i
for each k
if $A[i,k] \notin F_k$
list all j w. $A[i,k] = B[k,j]$
& increment $C[i,j]$.

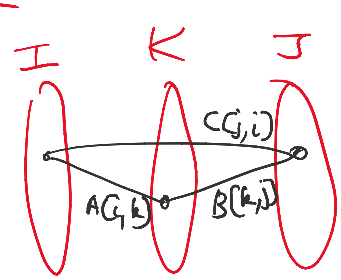
$\le \frac{n}{r}$ such j's

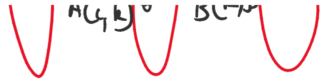
$\Rightarrow O(n^2 \cdot \frac{n}{r})$

total $\Rightarrow$ $O(rn^{\omega} + \frac{n^3}{r}) \Rightarrow O(n^{\frac{3+\omega}{2}})$. □

ZWT: (tripartite graph)
AE version

Given $n \times n$ matrices A, B, C,
$\forall i, j$
decide $\exists k$,





$\forall\, i, j$

decide $\exists k$,

$A(i,k) + B(k,j) = -C(j,i)$

AE-ZWT $\equiv$ ZWT

Similarly, #AE-ZWT

Reducing #AE-ZWT to AE-ZWT

Suppose AE-ZWT could be solved in $O(n^{3-\delta})$ time. by alg'm $\mathcal{A}$

To solve #AE-ZWT:

$\forall i, j$, let $W_{ij} = \{k : A(i,k) + B(k,j) = -C(j,i)\}$ ← witnesses

Few witnesses case: $|W_{ij}| \le n^{0.9}$

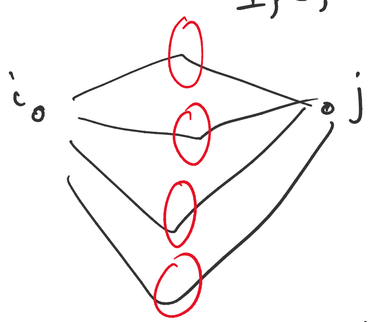
Will find all witnesses.

Warm-up: if witness is unique ($|W_{ij}| = 1$),

for $\ell = 1, \dots, \log n$,

run $\mathcal{A}$ on subgraph

$I, J, \{k \in K : \ell^{th}$ bit of k is $1\}$



Randomly partition K into $n^{0.9}$ groups of size $n^{0.1}$

I into " "

J into " "

for each fixed witness k,

Pr(k is unique in its group)

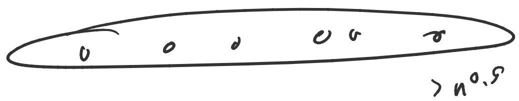
$\le \left(1 - \frac{1}{n^{0.9}}\right)^{n^{0.9}} = \Omega(1).$

$$\Rightarrow \tilde{O}\left((n^{0.9})^3 \cdot (n^{0.1})^{3-\delta}\right)$$
$$= \tilde{O}(n^{3-0.1\delta}).$$

Many witnesses case: $|W_{ij}| > n^{0.9}$.

Pick rand subset R of size $n^{0.1}\log n$

$\Rightarrow$ R hits all such W_{ij}

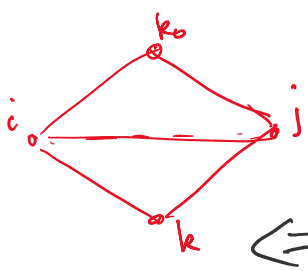


Fix $k_0 \in R$.

if $k_0 \in W_{ij}$,

count # of k s.t.

$$A(i,k) + B[k,j] = -C(j,i)$$
$$= A(i,k_0) + B[k_0,j]$$



$$\Longleftrightarrow \underbrace{A(i,k) - A(i,k_0)}_{A'(i,k)} = \underbrace{B[k_0,j] - B[k,j]}_{B'(k,j)}$$

Fredman's trick

$a+b = a'+b'$
$a - a' = b' - b$

this is Equality Prod!!

$$\Rightarrow \tilde{O}(n^{0.1} \cdot n^{2.69}) = \tilde{O}(n^{2.79}).$$

$\square$

Open Qs:

4SUM vs. #4SUM?

OV vs. #OV?

SETH vs. #SETH?

Boolean matrix mult vs. 0-1 matrix mult.