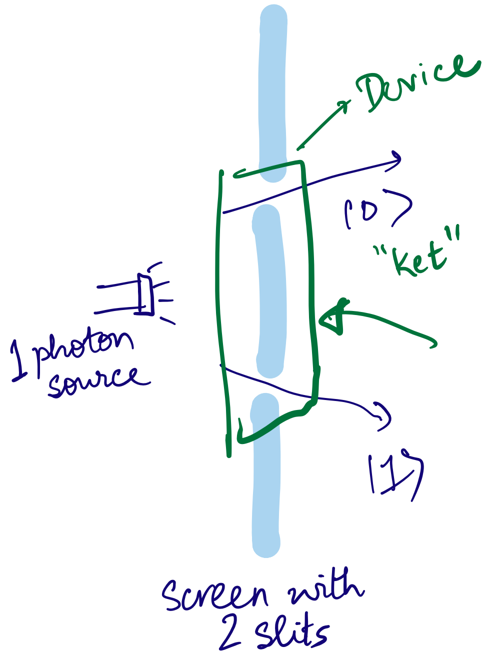
L

LECTURE - 2

$\boxed{\text{Origins of Q.M.}}$



Law #1 : If a quantum system / particle can be in state $|0\rangle$ or state $|1\rangle$ it can also be in a "superposition" state

particle/qubit

$$|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$$

"amplitude" on 0 "amplitude" on 1

where $|\alpha|^2 + |\beta|^2 = 1$, $(\alpha, \beta) \in \mathbb{C}^2$.

$= a + bi$

$|\alpha|^2 = a^2 + b^2$

$|0\rangle$ and $|1\rangle$ are "basis" states.

Examples.

"0.8" amplitude on $|0\rangle$, "0.6" amplitude on $|1\rangle$

$$0.8|0\rangle + 0.6|1\rangle.$$

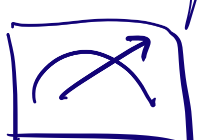
Probability of detector saying "0" will be $0.8^2 = 0.64$

$$0.8|0\rangle - 0.6|1\rangle$$

$$i|0\rangle + 0|1\rangle$$

$|i|^2 = 1$ $\quad$ $|0|^2 = 0.$

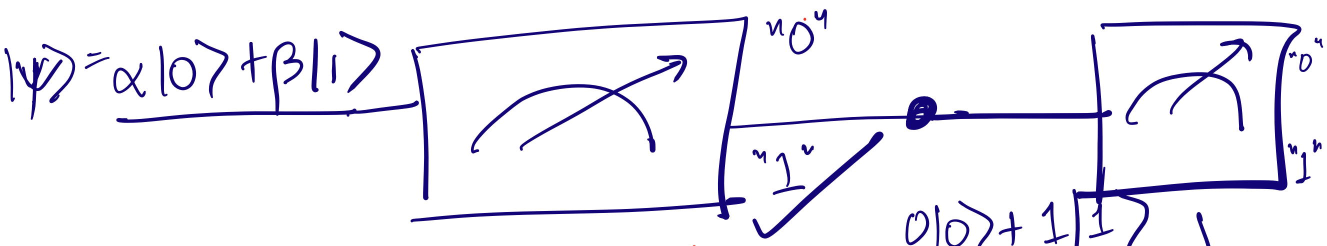
Law #2. for a particle with amplitude α on $|0\rangle$, β on $|1\rangle$ if you "measure" this particle,



"0" or "1" outcome

then you obtain "0" w.p. $|\alpha|^2$

"1" w.p. $|\beta|^2$



will always say "1".

$\alpha = 0.8$
$\beta = 0.6$

64% prob. of "0"
36% prob. of "1"

$\alpha = 0.8$
$\beta = -0.6$

same probabilities

$\boxed{\text{Qubit}}$ is basic unit of quantum information.

$$|\psi\rangle = \alpha|0\rangle + \beta|1\rangle, \quad \alpha, \beta \in \mathbb{C}^2$$

and $|\alpha|^2 + |\beta|^2 = 1$

"ket" notation
$|\;\rangle$

$$\begin{bmatrix} \alpha \\ \beta \end{bmatrix}$$

complex entries

Qubit is a 2-D unit vector in $\mathbb{C}^2$.

$|0\rangle$ and $|1\rangle$: basis vectors

$= 1|0\rangle + 0|1\rangle$ $\quad$ $= 0|0\rangle + 1|1\rangle$

$\begin{bmatrix} 1 \\ 0 \end{bmatrix}$, $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$ are basis of $\mathbb{C}^2$

These are orthogonal to each other.
orthonormal

$\{|0\rangle, |1\rangle\}$ is called "computational" basis.

$$\begin{bmatrix}\alpha\\ \beta\end{bmatrix} = \alpha\begin{bmatrix}1\\0\end{bmatrix} + \beta\begin{bmatrix}0\\1\end{bmatrix}$$

$$= \alpha'\begin{bmatrix}1\\1\end{bmatrix} + \beta'\begin{bmatrix}1\\-1\end{bmatrix}$$

$$\begin{bmatrix}\alpha\\ \beta\end{bmatrix} = \begin{bmatrix}\alpha'+\beta'\\ \alpha'-\beta'\end{bmatrix} \implies \begin{array}{l}\alpha' = (\alpha+\beta)/2\\ \beta' = (\alpha-\beta)/2\end{array}$$

$$\begin{bmatrix}\alpha\\ \beta\end{bmatrix} = \left(\frac{\alpha+\beta}{2}\right)\begin{bmatrix}1\\1\end{bmatrix} + \left(\frac{\alpha-\beta}{2}\right)\begin{bmatrix}1\\-1\end{bmatrix}$$

$$|+\rangle = \frac{1}{\sqrt{2}}|0\rangle + \frac{1}{\sqrt{2}}|1\rangle \qquad \begin{bmatrix}\frac{1}{\sqrt{2}}\\ \frac{1}{\sqrt{2}}\end{bmatrix}$$

$$|-\rangle = \frac{1}{\sqrt{2}}|0\rangle - \frac{1}{\sqrt{2}}|1\rangle \qquad \begin{bmatrix}\frac{1}{\sqrt{2}}\\ -\frac{1}{\sqrt{2}}\end{bmatrix}$$

$\{|+\rangle, |-\rangle\}$ is "Hadamard" basis.

Quantum Operations

"Phase shift"

$$\frac{|0\rangle + |1\rangle}{\sqrt{2}} \longrightarrow \frac{|0\rangle + e^{i\theta}|1\rangle}{\sqrt{2}}$$

$\theta = \pi$

$$\frac{|0\rangle - |1\rangle}{2}$$

$|e^{i\theta}|^2 = 1$

$$\alpha|0\rangle + \beta|1\rangle \longrightarrow \alpha|0\rangle + e^{i\theta}\beta|1\rangle$$

$$\begin{bmatrix} 1 & 0 \\ 0 & e^{i\theta} \end{bmatrix} \begin{bmatrix} \alpha \\ \beta \end{bmatrix} = \begin{bmatrix} \alpha \\ e^{i\theta}\beta \end{bmatrix}$$

shifts phase on $|1\rangle$

$\theta = \pi \quad e^{i\theta} = -1 \qquad \alpha|0\rangle + \beta|1\rangle \longrightarrow \alpha|0\rangle - \beta|1\rangle$

$\theta = \pi/2 \quad e^{i\theta} = i \qquad \alpha|0\rangle + \beta|1\rangle \longrightarrow \alpha|0\rangle + i\beta|1\rangle$

$$\rightarrow \begin{bmatrix} e^{i\theta} & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \alpha \\ \beta \end{bmatrix} = \begin{bmatrix} e^{i\theta}\alpha \\ \beta \end{bmatrix}$$

shift the phase on $|0\rangle$

"Bit flip"

$$|0\rangle \longrightarrow |1\rangle, \qquad |1\rangle \longrightarrow |0\rangle$$

"Hadamard transform"

$$H(|0\rangle) \longrightarrow |+\rangle = \frac{|0\rangle + |1\rangle}{\sqrt{2}}$$

$$H(|1\rangle) \longrightarrow |-\rangle = \frac{|0\rangle - |1\rangle}{\sqrt{2}}$$

$$H\left(\alpha|0\rangle + \beta|1\rangle\right) \longrightarrow \alpha H(|0\rangle) + \beta H(|1\rangle)$$

$$H(|+\rangle) = H\left(\frac{|0\rangle + |1\rangle}{\sqrt{2}}\right)$$

$$= \frac{H(|0\rangle)}{\sqrt{2}} + \frac{H(|1\rangle)}{\sqrt{2}}$$

$$= \frac{|0\rangle + \cancel{|1\rangle}}{2} + \frac{|0\rangle \cancel{- |1\rangle}}{2}$$

$$= |0\rangle.$$

$$H(|-\rangle) = |1\rangle$$

Apply <u>any</u> linear transform that preserves "energy"

$$|0\rangle \rightarrow U_{00}|0\rangle + U_{01}|1\rangle$$

$$|1\rangle \rightarrow U_{10}|1\rangle + U_{11}|1\rangle$$

$$U = \begin{bmatrix} U_{00} & U_{10} \\ U_{01} & U_{11} \end{bmatrix}$$

$U(|0\rangle) \longrightarrow$ written as $U|0\rangle$

$$\begin{bmatrix} U_{00} & U_{10} \\ U_{01} & U_{11} \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} U_{00} \\ U_{01} \end{bmatrix}$$

$$= U_{00}|0\rangle + U_{01}|1\rangle$$

$$U|1\rangle \quad \begin{bmatrix} U_{00} & U_{10} \\ U_{01} & U_{11} \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} U_{10} \\ U_{11} \end{bmatrix}$$

$$= U_{10}|0\rangle + U_{11}|1\rangle.$$

$$U|\psi\rangle \rightarrow \begin{bmatrix} U_{00} & U_{10} \\ U_{01} & U_{11} \end{bmatrix} \begin{bmatrix} \alpha \\ \beta \end{bmatrix} = \begin{bmatrix} (U_{00}\alpha + U_{10}\beta) \\ (U_{01}\alpha + U_{11}\beta) \end{bmatrix}$$

Recall: We need, for every qubit

$|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$ that

$$1 = |\alpha|^2 + |\beta|^2 = \alpha\alpha^* + \beta\beta^*$$

$$= \underbrace{\begin{bmatrix} \alpha^* & \beta^* \end{bmatrix}}_{(|\psi\rangle)^\dagger} \underbrace{\begin{bmatrix} \alpha \\ \beta \end{bmatrix}}_{|\psi\rangle}$$

$$= (|\psi\rangle)^\dagger \; |\psi\rangle$$

For any $|\psi\rangle$, $(|\psi\rangle)^\dagger = \langle\psi|$ "bra"-"psi".

$|\,\rangle$: ket notation

$\langle\,|$: bra notation

For every qubit $|\psi\rangle$,

$$\langle\psi|\psi\rangle = 1$$

Now lets say we applied $U|\psi\rangle$

$U = \begin{bmatrix} U_{00} & U_{10} \\ U_{01} & U_{11} \end{bmatrix}$ to obtain $|\phi\rangle$

$$|\phi\rangle = U|\psi\rangle \quad \text{where } U = \begin{bmatrix} U_{00} & U_{10} \\ U_{01} & U_{11} \end{bmatrix}$$

$$|\psi\rangle = \begin{bmatrix} \alpha \\ \beta \end{bmatrix}$$

Then, as long as $|\psi\rangle$ is a qubit

(i.e. $|\alpha|^2 + |\beta|^2 = 1$
equivalently $\langle\psi|\psi\rangle = 1$)

$\phi = U|\psi\rangle$ must also be a qubit.

i.e. $\langle\phi|\phi\rangle = 1$

$$\langle\phi|\phi\rangle = (|\phi\rangle)^\dagger |\phi\rangle$$

$$= (U|\psi\rangle)^\dagger (U|\psi\rangle)$$

$$= \langle\psi| \underbrace{U^\dagger_{2\times 2} U_{2\times 2}}_{= M_{2\times 2}} |\psi\rangle$$

$$1 = \langle\psi|M|\psi\rangle$$

For every qubit $|\psi\rangle$,

$$\langle \psi | M | \psi \rangle = 1$$

This means $\forall\, \alpha, \beta$ s.t. $|\alpha|^2 + |\beta|^2 = 1$,

$$\begin{bmatrix} \alpha^* & \beta^* \end{bmatrix} \begin{bmatrix} M_{00} & M_{10} \\ M_{01} & M_{11} \end{bmatrix} \begin{bmatrix} \alpha \\ \beta \end{bmatrix} = 1.$$

i.e.

$$\begin{bmatrix} \alpha^* M_{00} + \beta^* M_{01} & \alpha^* M_{10} + \beta^* M_{11} \end{bmatrix} \begin{bmatrix} \alpha \\ \beta \end{bmatrix} = 1$$

i.e.

$$\underbrace{\alpha\alpha^*}_{|\alpha|^2} \overset{1}{M_{00}} + \alpha\beta^* \overset{0}{M_{01}} + \alpha^*\beta \overset{0}{M_{10}} + \underbrace{\beta\beta^*}_{|\beta|^2} \overset{1}{M_{11}} = 1$$

Want this to be 1 NO MATTER what α and β are.

$$M = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I_{2\times 2}$$

i.e. $U^\dagger U = I$

Def. A matrix U which satisfies $U^\dagger U = I$ is called a UNITARY MATRIX.

Properties of unitary matrices

$$U^\dagger U = I \iff U^\dagger = U^{-1}$$

$\iff$ columns of U are ~~tho~~ orthonormal vectors

U is $n \times n$

$$(U^\dagger U)_{ij} = \sum_{k \in [n]} (U^\dagger)_{ik} \cdot U_{kj}$$

$$= \sum_{k \in [n]} \underbrace{U^*_{ki}}_{i^{th} \text{ column}} \cdot \underbrace{U_{kj}}_{j^{th} \text{ column}}$$

$(i,j)^{th}$ entry of $\underset{= I}{U^\dagger U}$ is 0 when $i \neq j$

1 when $i = j$

$$\boxed{U \text{ is called Hermitian if additionally } U = U^\dagger}$$

Multi-Qubit Systems

2 qubits

"tensor product" notation

both $|0\rangle \rightarrow |00\rangle$ $\quad |0\rangle \otimes |0\rangle$

first $|0\rangle$, second $|1\rangle \rightarrow |01\rangle$ $\quad |0\rangle \otimes |1\rangle$

$\rightarrow |10\rangle$ $\quad |1\rangle \otimes |0\rangle$

$\rightarrow |11\rangle$ $\quad |1\rangle \otimes |1\rangle$

$\{|00\rangle, |01\rangle, |10\rangle, |11\rangle\}$
are basis states of a 2-qubit system

$$|\phi_0\rangle = \alpha_0 |0\rangle + \beta_0 |1\rangle$$
$$|\phi_1\rangle = \alpha_1 |0\rangle + \beta_1 |1\rangle$$

joint state

$$|\psi\rangle = |\phi_0\rangle \otimes |\phi_1\rangle$$
$$= (\alpha_0 |0\rangle + \beta_0 |1\rangle) \otimes (\alpha_1 |0\rangle + \beta_1 |1\rangle)$$
$$= (\alpha_0\alpha_1 (|0\rangle \otimes |0\rangle) + \alpha_0\beta_1 (|0\rangle \otimes |1\rangle) \ldots\ldots$$
$$= \alpha_0\alpha_1 |00\rangle + \alpha_0\beta_1 |01\rangle + \beta_0\alpha_1 |10\rangle + \beta_0\beta_1 |11\rangle$$

$\boxed{\text{H.W.}}$ Check that $|\alpha_0\alpha_1|^2 + |\alpha_0\beta_1|^2 + |\beta_0\alpha_1|^2 + |\beta_0\beta_1|^2 = 1$

$$|\psi\rangle = \alpha_{00}|00\rangle + \alpha_{01}|01\rangle + \alpha_{10}|10\rangle + \alpha_{11}|11\rangle$$

"Measure" only the first qubit in the computational basis

Square the amplitudes on terms that have 0 in the first position.

$\left(|\alpha_{00}|^2 + |\alpha_{01}|^2\right)$: probability with which you observe a "0".

state collapse to $$\frac{\alpha_{00}|00\rangle + \alpha_{01}|01\rangle}{\sqrt{|\alpha_{00}|^2 + |\alpha_{01}|^2}}$$

$$= |0\rangle \otimes \frac{\alpha_{00}|0\rangle + \alpha_{01}|1\rangle}{\sqrt{|\alpha_{00}|^2 + |\alpha_{01}|^2}}$$