LECTURE-8.

Today

* QFT continued
* Mixed states

Announcement: HW1 is out on Piazza
(also a gradescope signup link on the piazza post).

Implementing Quantum Fourier Transform.

* QFT over $\mathbb{Z}_2^n$: $H^{\otimes n}$

inner product of bitstrings σ and x mod 2.

$$\sum_{x \in \{0,1\}^n} \alpha_x |x\rangle \equiv \sum_{\sigma \in \{0,1\}^n} \sum_{x \in \{0,1\}^n} (-1)^{\sigma x} \alpha_x |x_\sigma\rangle.$$

$$\Big\downarrow \boxed{QFT}$$

$$\sum_{\sigma \in \{0,1\}^n} \sum_{x \in \{0,1\}^n} (-1)^{\sigma x} \alpha_x |\sigma\rangle.$$

$$H^{\otimes n}\Big(\sum_{x \in \{0,1\}^n} \alpha_x |x\rangle\Big) = \sum_{\sigma \in \{0,1\}^n} \sum_{x \in \{0,1\}^n} (-1)^{\sigma x} \alpha_x |\sigma\rangle.$$

* QFT over $\mathbb{Z}_N$ for $N > 2$.

N.

$$\sum_{x\in[0,N-1]} \alpha_x |x\rangle = \sum_{\sigma\in[0,N-1]} \sum_{x\in[0,N-1]} \omega^{-\sigma x} \alpha_x |x_\sigma\rangle$$

ω: N^{th} root of unity

σx: product of σ, x mod N.

$$\Big\downarrow \text{ QFT}$$

$$\sum_{\sigma\in[0,N-1]} \sum_{x\in[0,N-1]} \omega^{-\sigma x} \alpha_x |\sigma\rangle$$

MAP.

$$|x\rangle \longrightarrow \sum_{\sigma\in[0,N-1)} \omega^{-\sigma x} |\sigma\rangle.$$

last time, example:

$N = 2^4 = 16$, $x \in [0, 15] = x_3 x_2 x_1 x_0$.

$$|x\rangle \longrightarrow |0\rangle + \omega^{-x}|1\rangle + \omega^{-2x}|2\rangle \ldots\ldots + \omega^{-15x}|15\rangle$$

$= |x_3 x_2 x_1 x_0\rangle$.

$$= |0000\rangle + \omega^{-x}|0001\rangle + \omega^{-2x}|0010\rangle \ldots\ldots + \omega^{-15x}|1111\rangle$$

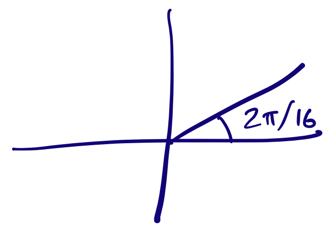
$$= \left(\frac{|0\rangle + \omega^{-8x}|1\rangle}{\sqrt{2}}\right) \otimes \left(\frac{|0\rangle + \omega^{-4x}|1\rangle}{\sqrt{2}}\right) \otimes \left(\frac{|0\rangle + \omega^{-2x}|1\rangle}{\sqrt{2}}\right) \otimes \left(\frac{|0\rangle + \omega^{-x}|1\rangle}{\sqrt{2}}\right)$$

$$= |\gamma_3\rangle \otimes |\gamma_2\rangle \otimes |\gamma_1\rangle \otimes |\gamma_0\rangle$$

$$|\gamma_3\rangle = \frac{|0\rangle + \omega^{-8x}|1\rangle}{\sqrt{2}}$$

$$= \frac{|0\rangle + (\omega^{-8})^{x}|1\rangle}{\sqrt{2}}$$

$\omega^{16} = 1$



$$= \frac{|0\rangle + (-1)^{x}|1\rangle}{\sqrt{2}}$$

$$= \frac{|0\rangle + (-1)^{x_0}|1\rangle}{\sqrt{2}}$$

ω^8 forms angle $\frac{2\pi}{16} \times 8 = \pi$

$$|\gamma_3\rangle = \frac{|0\rangle + (-1)^{x_0}|1\rangle}{\sqrt{2}}$$

$$= H(|x_0\rangle).$$

$\omega^{8} = -1.$

$\omega^{-8} = \frac{1}{\omega^8} = -1.$

$$|\gamma_2\rangle = \frac{|0\rangle + \omega^{-4x}|1\rangle}{\sqrt{2}}$$

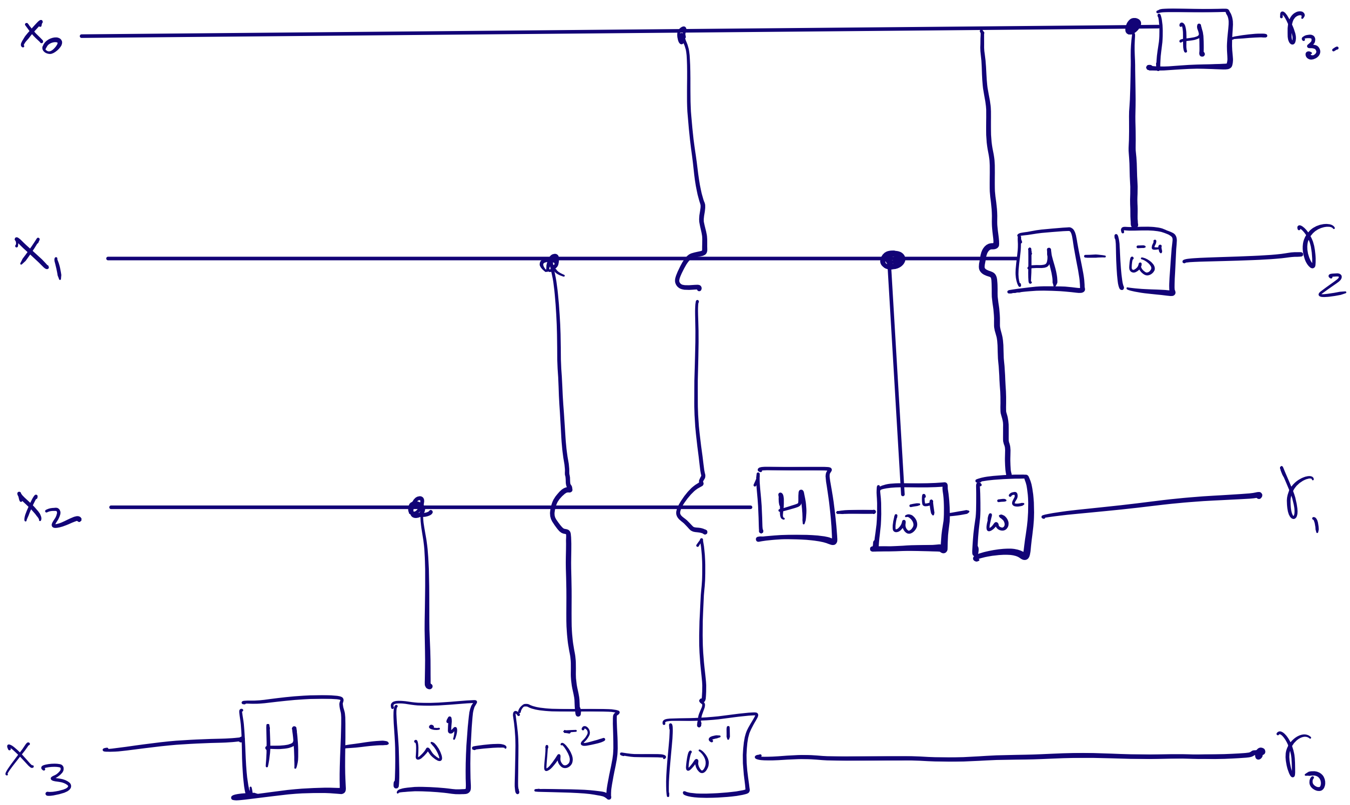
$$\omega^{-4x} = \omega^{-4(8x_3 + 4x_2 + 2x_1 + x_0)}$$

$$= \underbrace{\cancel{\omega^{-32x_3}}}_{1} \cdot \underbrace{\cancel{\omega^{-16x_2}}}_{1} \cdot \omega^{-8x_1} \cdot \omega^{-4x_0}$$

$$= (-1)^{x_1} \cdot (\omega^{-4})^{x_0}$$

$$|\gamma_2\rangle = \frac{|0\rangle + (-1)^{x_1} \cdot (\omega^{-4})^{x_0}|1\rangle}{\sqrt{2}}$$

QFT over $\mathbb{Z}_{2^n}$



where $\omega^{-4} = \begin{bmatrix} 1 & 0 \\ 0 & \omega^{-4} \end{bmatrix}$ $\quad \omega^{-2} = \begin{bmatrix} 1 & 0 \\ 0 & \omega^{-2} \end{bmatrix}$

$$|y_1\rangle = \frac{|0\rangle + \omega^{-2x}|1\rangle}{\sqrt{2}}$$

$$= \frac{|0\rangle + \left(\omega^{-8x_2} \cdot \omega^{-4x_1} \cdot \omega^{-2x_0}\right)|1\rangle}{\sqrt{2}}$$

$$= |0\rangle + \frac{(-1)^{x_2} \cdot (\omega^{-4})^{x_1} \cdot (\omega^{-2})^{x_0}|1\rangle}{\sqrt{2}}$$

$\mathbb{Z}_N$ where N is a power of 2.

$\mathbb{Z}_p$ where p is an arbitrary positive integer.

$$\underset{\in \mathbb{Z}_p}{|x\rangle} \xrightarrow{QFT_{\mathbb{Z}_p}} \sum_{\sigma \in \mathbb{Z}_p} \omega^{-\sigma x} |\sigma\rangle .$$

$$\mathbb{Z}_2^n \quad , \quad \mathbb{Z}_p .$$

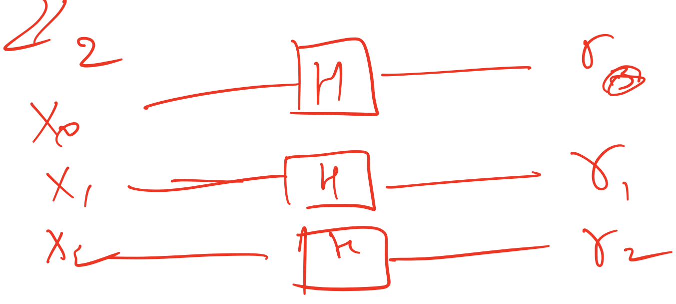
QFT over $\mathbb{Z}_p^n$.

$\langle \sigma, x \rangle \bmod p$.

$$\underset{\in \mathbb{Z}_p^n}{|x_1, x_2 \dots x_n\rangle} \xrightarrow{QFT_{\mathbb{Z}_p^n}} \sum_{\sigma \in \mathbb{Z}_p^n} \omega^{-\sigma x} |\sigma\rangle$$

p^{th} root of unity

QFT over $\mathbb{Z}_2^n$



$$\sum_{x_1 \ldots x_n \in [0,p-1]^n} \alpha_{x_1 x_2 \ldots x_n} |x_1\rangle |x_2\rangle \ldots |x_n\rangle$$

$$\Big\downarrow \text{ QFT over } \mathbb{Z}_p^n$$

$$\sum_{\sigma_1 \ldots \sigma_n \in [0,p-1]^n} \left(\sum_{x_1 x_2 \ldots x_n \in [0,p-1]^n} \omega^{-\langle \sigma, x \rangle \bmod p} \alpha_{x_1 \ldots x_n} \right) |\sigma_1 \sigma_2 \ldots \sigma_n\rangle$$

For example.

$$\sum_{x_1 x_2} \alpha_{x_1 x_2} |x_1\rangle |x_2\rangle \qquad \mathbb{Z}_p^2.$$

$$\Big\downarrow \text{ QFT over } \mathbb{Z}_p^2$$

$$\sum_{\sigma_1 \sigma_2} \sum_{x_1 x_2} \omega^{-(\sigma_1 x_1 + \sigma_2 x_2)} \alpha_{x_1 x_2} |\sigma_1\rangle |\sigma_2\rangle$$

for example

$$\sum_{x \in [0,p-1]} |x\rangle |ax+b\rangle$$

f is a linear function $f(x) = (ax+b) \bmod p$.

$$\Big\downarrow \text{ QFT over } \mathbb{Z}_p^2$$

$$\sum_{\sigma_1 \sigma_2} \sum_{x} \omega^{-(\sigma_1 x + \sigma_2 (ax+b))} |\sigma_1 \sigma_2\rangle.$$

$$= \sum_{\sigma_1 \sigma_2} \sum_{x} \omega^{-x(\sigma_1 + \sigma_2 a)} \cdot \omega^{-\sigma_2 b} |\sigma_1 \sigma_2\rangle.$$

when $\sigma_1 + \sigma_2 a = 0$ then $\omega^{-x(\sigma_1 + \sigma_2 a)} \neq 0$.

Say $(\sigma_1 + \sigma_2 a) = 1$

$$\sum_x \omega^{-x(\sigma_1+\sigma_2 a)} = \sum_x \omega^{-x}$$

$$= 1 + \omega^{-1} + \omega^{-2} \dots \omega^{-(p-1)}$$

$$= \frac{\omega^{-p} - 1}{\omega^{-1} - 1} = 0.$$

Example.

$$\sum_x \omega^{cx} |x\rangle. \quad \text{Find } c.$$

$$\downarrow \text{QFT} \qquad = \sum_\sigma \sum_x \omega^{cx} \omega^{-\sigma x} |x_\sigma\rangle$$

$$\sum_\sigma \sum_x \omega^{cx} \cdot \omega^{-\sigma x} |\sigma\rangle.$$

$$= \sum_\sigma \left(\sum_x \omega^{(c-\sigma)x} \right) |\sigma\rangle.$$

Amplitude of state, is 0 on any σ where $(\sigma - c) \neq 0$

Amplitude is non-zero on σ where $\sigma - c = 0$

MIXED STATE

Pure state. $|\psi\rangle = \sum_{x \in \ldots} \alpha_x |x\rangle$.

$|0\rangle$ w.p. $\frac{1}{2}$

$|1\rangle$ w.p. $\frac{1}{2}$

$\Big\} \longrightarrow$ output the result.

Resulting state: $\left\{ \left(\frac{1}{2}, |0\rangle\right), \left(\frac{1}{2}, |1\rangle\right) \right\}$

$|+\rangle$ w.p. 0.99

$|0\rangle$ w.p. 0.005

$|1\rangle$ w.p. 0.005.

Resulting state $\{ (0.99, |+\rangle), (0.005, |0\rangle), (0.005, |1\rangle) \}$.

Density matrix.

Square matrix ρ is a density matrix if.

- ρ is positive semi-definite
  - Hermitian, $\rho^\dagger = \rho$ Eg $\begin{bmatrix} 1 & 1+i \\ 1-i & 1 \end{bmatrix}$
  - $\langle x | \rho | x \rangle \geq 0 \quad \forall x \in \mathbb{C}^n$.
- $\text{Tr}[\rho] = 1$.

Density matrices.

For pure state $|\psi\rangle$, $\rho = |\psi\rangle\langle\psi|$.

$$|\psi\rangle = \begin{bmatrix} \alpha_0 \\ \alpha_1 \\ \vdots \\ \alpha_N \end{bmatrix} \in \mathbb{C}^n$$

$$|\psi\rangle\langle\psi| = \begin{bmatrix} \alpha_0 \\ \alpha_1 \\ \vdots \\ \alpha_N \end{bmatrix} \begin{bmatrix} \alpha_0^* & \alpha_1^* & \cdots & \alpha_N^* \end{bmatrix}$$

$$= \begin{bmatrix} \alpha_0\alpha_0^* & \alpha_0\alpha_1^* & \cdots & \alpha_0\alpha_N^* \\ \vdots & & & \\ \alpha_N\alpha_0^* & \cdots & \cdots & \alpha_N\alpha_N^* \end{bmatrix}_{N \times N}$$

✓ $Tr = 1$.

✓ Hermitian

HW: $\langle x|\rho|x\rangle \geq 0$.

Mixed states.

For every mixed state, there is exactly one density matrix.

$$\{ (p_1, |\psi_1\rangle), (p_2, |\psi_2\rangle) \ldots \ldots \}_{m \text{ terms}}$$

$$\rho = \sum_{i \in [m]} p_i |\psi_i\rangle\langle\psi_i|$$

✓ $\mathrm{Tr} = 1$

Linear comb. of p.s.d. matrices

For every density matrix, is there a unique division into $\{(p_i, |\psi_i\rangle)\}$? No.