HW 1

Due on **Wednesday, September 28, 2022 at 10am**

CS 598: Computational Social Choice, Fall 2022 Version: **1**

1 (100 PTS.) Arrow-Debreu Model (Fairness + PO)

Recollect the linear Arrow-Debreu market model. There are n agents, and m *divisible* goods. Agent i has a utility of u_{ij} for one unit of good j. Each agent i comes with an endowment of w_{ij} units of good j. A price vector p and an allocation x is at an equilirbium, iff,

- All items are sold, i.e., $\sum_{i\in[n]} x_{ij} = 1$, for all $j \in [m]$.
- All agents spend $\sum_{j\in[m]} w_{ij}p_j$ units of money on bang-per buck goods, i.e., $x_{ij} > 0$ only if $u_{ij}/p_j \geq u_{ik}/p_k$ for all $k \in [m]$, and $\sum_{j\in[m]} x_{ij}p_j = \sum_{j\in[m]} w_{ij}p_j$.

Prove that, $\langle p, x\rangle$ satisfies,

- *Weighted Envy-Freeness:* For all agents i, and i', we have $\frac{u_i(x_i)}{\sum_{j\in[m]} w_{ij}p_j} \geq \frac{u_i(x_{i'})}{\sum_{j\in[m]} w_{i'j}p_j}$.[1]
- *Pareto optimal:* There exists no other allocation y, such that $u_i(y_i) \geq u_i(x_i)$ for all $i \in [n]$ with at least one strict inequality.

[1] Recall that $u_i(x_i) = \sum_{j\in[m]} u_{ij}x_{ij}$ as we are in the linear model.

2 (100 PTS.) Perturbation in Markets

Most of the flow based algorithms in market equilibrium involve computation on the equality network N_p. In worst case, N_p can contain $\mathcal{O}(nm)$ edges (n is the number of agents and m is the number of goods). Often times, this could be very expensive. Sparse equality networks facilitate faster algorithms. In this exercise, we investigate when equality networks can be sparse.

2.A. (75 PTS.) Let $C = i_1 - j_1 - i_2 - j_2 - \cdots - i_k - j_k - i_1$ be a cycle in the equality network N_p. Prove that $\prod_{\ell \in [k]} u_{i_{\ell+1}, j_\ell} = \prod_{\ell \in [k]} u_{i_\ell, j_\ell}$ (interpret $k+1$ as 1).

2.B. (25 PTS.) Choose an infinitesimally small $\varepsilon \geq 0$. Perturb every u_{ij} to $u'_{ij} = u_{ij} q_{ij}^{\varepsilon}$ where q_{ij}'s are distinct primes. Prove that any prices, the equality network, defined w.r.t. the perturbed utilities is always acyclic.

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3 (100 PTS.) Eliminating Path Violators

Recollect the market-based algorithm for computing EF1 + PO. Recall that in the allocation update phase, we eliminated all path violators. Let i_0^t be a least spender at time t, and i_ℓ^t be a closest path violator. Let $i_0^t \rightarrow j_1^t \rightarrow i_1^t \rightarrow j_2^t \rightarrow i_2^t \ldots \rightarrow j_\ell^t \rightarrow i_\ell^t$ be the alternating shortest path from i_0^t to i_ℓ^t, with the blue edges indicating MBB edges and black edges indicating allocation edges. Recall that in the allocation update phase, as long as i_ℓ^t is a path-violator, we transfer g_ℓ^t from i_ℓ^t to $i_{\ell-1}^t$ and decrement ℓ. Once this cascade of transfers stops along the current alternating path, we again check for new path-violators and continue the same procedure. In this exercise, we want to show that we can have at most polynomial number of such *consecutive* transfers until there is a change in the identity of the least spender or all path violators are eliminated. We will make use of the following notation to prove our claim. For each agent i reachable from a least spender i_0^t at time t via an alternating path,

- define $\mathit{level}^t(i)$ to the half the length of the shortest alternating path from i_0^t to i_k^t (note that alternating path lengths are always even). For all agents i not reachable from i_0^t at time t, define $\mathit{level}^t(i) = n$.
- We say that g is critical to agent i at time t, if $g \in x_i^t$ and there is an alternating shortest path from i_0^t to i that also contains g. Define G_i^t to be the set of critical goods of agent i at time t.

Prove that with each tansfer operation (as elaborated above), the value of $\sum_{i \in [n]} \big(m(n - \mathit{level}^t(i)) + G_i^t \big)$ decreases. Use this to show that after poly(n, m) many transfers, either the identity of the least spender changes or all path violators are eliminated.