CS 580: Topics on Algorithmic Game Theory, Fall 2026
HW 1 (due on Wednesday, Sept 23rd at 11:59pm CST)

Instructions:

1. We will grade this assignment out of a total of 30 points.
2. You can work on any homework in groups of ($\leq$) two. Submit only one assignment per group. First submit your solutions on Gradescope, and you can add your group member after submission.
3. If you discuss a problem with another group then write the names of the other group's members at the beginning of the answer for that problem.
4. Please type your solutions if possible in Latex or doc whichever is suitable, and submit on Gradescope.
5. Even if you are not able to solve a problem completely, do submit whatever you have. Partial proofs, high-level ideas, examples, and so on.
6. Except where otherwise noted, you may refer to lecture slides/notes. You cannot refer to textbooks, handouts, or research papers that have not been listed. If you do use any approved sources, make sure you **cite them appropriately**, and make sure to **write in your own words**.
7. No late assignments will be accepted.
8. By AGT book we mean the following book: Algorithmic Game Theory (edited) by Nisan, Roughgarden, Tardos and Vazirani. Its free online version is available at Prof. Vijay V. Vazirani's webpage.

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1. (**Recommendation: Try doing this by yourself without using LLM tools. If you end up using LLM/AI tools, write the final answer yourself and cite your sources.**)

   Consider a market $\mathcal{M}$ with n agents and m divisible goods, where supply of good j is q_j and the valuation of agent i is defined by a monotonically non-decreasing concave function $V_i : \mathbb{R}_+^m \to \mathbb{R}_+$. A *competitive equilibrium with equal income (CEEI)* of such a market is a pair (X, p) where p is a price vector $(p_1, \ldots, p_m)$, p_j is the price-per-unit of good j and $X = (X_1, \ldots, X_n)$ is an allocation of goods to agents s.t.,

   - **Optimal Bundle.** For each agent i, $X_i \in \text{argmax}_{Y \geq 0 : \sum_j p_j Y_j \leq 1} V_i(Y)$.
   - **Demand equals Supply.** For each good j, $\sum_{i \in [n]} X_{ij} \leq q_j$, and whenever $p_j > 0$ we have $\sum_{i \in [n]} X_{ij} = q_j$.

(a) (5 points) Show that it is without loss of generality to assume that $q_j = 1$ for all goods j, that is, to assume that the supply of every good is one. Formally, come up with another market $\mathcal{M}' = ([n], [m], (q'_j)_{j\in[m]}, (V'_i)_{i\in[n]})$ such that $q'_j = 1, \ \forall j \in [m]$, and show that a CEEI of $\mathcal{M}'$ can be mapped to a CEEI of $\mathcal{M}$.

(b) (5 points) Show that when the DPSV algorithm is executed to compute a CEEI of such a market, Event 2 of the algorithm can always be computed in polynomial time. That is, the value of α for which a new MBB edge appears between some agent $i \in A_D$ and some good $j \in G_F$, and the pair (i, j), can be computed in polynomial time. Here, α is the constant by which the prices of the goods in G_D are scaled in one iteration of the algorithm. Set A_D consists of agents with MBB edges to goods in G_D before α starts increasing, and G_F is the set of goods with demand = supply, meaning if the market consisted only of the subset (A_F, G_F), then the prices of goods in G_F and their allocation to A_F satisfy the requirements of a CEEI. (See Lecture 3 slides for further details)

2. (**Recommendation: Feel free to use LLM/AI tools. But again, write the answer in your own language making sure you understand the solution. And cite your sources.**)

   Design a polynomial-time algorithm to compute a Prop1+PO allocation. The algorithm should find a Prop1+PO allocation of m indivisible items, among n agents, each with an additive valuation function, namely $v_i(S) = \sum_{j\in S} v_{ij}$ for any $S \subseteq [m]$.

3. (**Experimental.**) Run the following experiment using any freely available LLM models, such as ChatGPT/Gemini/Calude etc.

   (a) Create a large enough instance, say at least 3 agents and 10 indivisible items, with additive valuations. Ask the LLM/AI model of your choice to compute a fair allocation, and ask why it is fair.

   Repeat the above experiment with at least five different instances and two LLM models, and report your findings in no more than two pages. Also, share your thoughts about how fair the outcome was according to you.

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4. (**Bonus question. Open-ended. Feel free to use LLM/AI tools and cite your sources.**) We know that EFX allocation exists for graphical valuations, and a more recent result extended it to multi-graph valuations. This falls into the regime of (p, q)-valuations. See the attached write-up generated by ChatGPT Pro for definitions and the state-of-the-art results; feel free to ask on Slack/Piazza if anything is not clear.

   Extend the existence result to any regime of (p, q) that is not known already.

Extra questions for exam practice, so do NOT use LLM/AI tools. Do not submit.

5. (*Short Questions*)

   (a) (2 points) Give an example with general monotone valuations where an EF1 allocation is not Prop1.

   (b) (3 points) Give an example with additive valuations where the round robin algorithm achieves better social welfare ($\sum_i V_i(A_i)$) than the envy-cycle-elimination algorithm under certain choices.

   (c) (2 points) Give an example with additive valuations where an EF1+PO allocation is not EFX.

   (d) (2 points) For additive valuation functions, we showed $MMS_i \leq \frac{v_i(M)}{n}$ for all agents i. Give an example with submodular valuation functions where this is not true, and in fact $MMS_i = v_i(M)$ for all agents i.

   (e) (1 point) Prove that if an α-MMS allocation exists for an instance, then an α-MMS+PO allocation also exists.

6. (*Algorithm Design*)

   (a) (5 points) Consider an instance with additive valuations where items are identically ordered. That is, there exists an ordering of items $g_1, g_2, \ldots, g_m$ such that for each agent i,

   $$v_{ig_1} \geq v_{ig_2} \geq \cdots \geq v_{ig_m}.$$

   Show that the *envy-cycle elimination* algorithm gives an EFX allocation when the items are considered in a particular order.

   (b) (5 points) For the case with additive valuation functions, where $v_i(M) = n$ for all agents i, show that when $v_{ij} \leq \epsilon$ for all agents i and goods j, an EF1+$(1-\epsilon)$-MMS allocation exists and can be computed in polynomial time.

Worked for 37s

The main extension is now to arbitrary multigraphs with cancelable valuations, including arbitrary additive valuations. This is a June 2026 preprint, so it is quite recent. Each good can matter to at most two agents, but a pair of agents can now share arbitrarily many goods. Afshinmehr et al., *EFX Allocations Exist on Multi-Graphs* .

There are two distinct directions of generalization: relaxing **which agents value which goods**, and relaxing **the valuation functions themselves**. The progression is:

Result	Structure of relevant goods	Valuations	Guarantee
Christodoulou–Fiat–Koutsoupias–Sgouritsa, EC 2023	Simple graph: each good matters to at most two agents; each pair shares at most one good	General monotone	Complete EFX allocation
Kaviani–Seddighin–Shahrezaei, 2024	Arbitrary multigraph	Restricted additive: $v_i(g) \in \{0, w_g\}$	Complete EFX allocation
Afshinmehr–Ashuri–Mahmoudkhan–Mehlhorn, December 2025	Triangle-free multigraph	General monotone	Complete EFX; pseudo-polynomial algorithm, polynomial for cancelable valuations
Afshinmehr–Ashuri–Mahmoudkhan–Mehlhorn–Shahrezaei, June 2026	**Arbitrary multigraph**	**Cancelable, a strict superclass of additive**	**Complete EFX in polynomial time**

Sources: the 2025 paper reviews the original result ; 2024 restricted-additive extension ; triangle-free extension ; 2026 multigraph extension .

The distinction matters: **the original theorem already allowed general monotone valuations**. Thus, the new arbitrary-multigraph theorem broadens the graph structure while imposing cancelability. It does not establish existence for general monotone valuations on all multigraphs.

For extensions allowing **more than two agents to value a good**, the useful notation is (p, q)-bounded valuations:

- Each good is relevant to at most p agents.
- Each pair of agents has at most q common relevant goods.

The original graphical setting is $(2, 1)$; the new multigraph result handles $(2, \infty)$ with cancelable valuations. Moving instead to $(\infty, 1)$—arbitrarily many agents per good, but pairwise overlap at most one—the results I found give **EF2X**, or **EFX with at most** $\lfloor n/2 \rfloor - 1$ **unallocated goods**, rather than complete exact EFX in general. Kaviani et al., 2024 . A June 2026 hypergraph paper further develops approximate EFX and EFkX guarantees, rather than resolving complete EFX existence in that broader setting. Kakatelis et al., *Almost EFX in Hypergraphs* .

Finally, these are **allocation** results: goods may be assigned to agents who do not value them. Requiring every good to go to an interested endpoint gives the stronger **EFX orientation** problem, for which existence can fail even on simple graphs. Zeng–Mehta, 2024 .