

CS 580: Algorithmic Game Theory, Fall 2025

HW 3 (due on Friday, 14th November at 11:59pm CST)

Instructions:

1. We will grade this assignment out of a total of 40 points.
2. You can work on any homework in groups of ($\leq$) two. Submit only one assignment per group. First submit your solutions on Gradescope and you can add your group member after submission.
3. If you discuss a problem with another group then write the names of the other group's members at the beginning of the answer for that problem.
4. Please type your solutions if possible in Latex or doc whichever is suitable, and submit on Gradescope.
5. Even if you are not able to solve a problem completely, do submit whatever you have. Partial proofs, high-level ideas, examples, and so on.
6. Except where otherwise noted, you may refer to lecture slides/notes. You cannot refer to textbooks, handouts, or research papers that have not been listed. If you do use any approved sources, make sure you **cite them appropriately**, and make sure to **write in your own words**.
7. No late assignments will be accepted.
8. By AGT book we mean the following book: Algorithmic Game Theory (edited) by Nisan, Roughgarden, Tardos and Vazirani. Its free online version is available at Prof. Vijay V. Vazirani's webpage.

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1. (*Routing games problem 1*)
 - (a) (5 points) Prove that if $\mathcal{C}$ is the set of cost functions of the form $c(x) = ax^2 + bx + c$ with $a, b, c \geq 0$, then the Pigou bound $\alpha(\mathcal{C})$ is $\frac{3\sqrt{3}}{3\sqrt{3}-2}$.
 - (b) (5 points) Give example of a potential game where the strategy profile achieving minimum potential does not give the minimum cost NE.
[Hint: Try cost-sharing game.]
 2. (10 points) This problem develops some theory about potential games; we talked about these while discussing selfish routing. We consider an abstract finite game with n players with finite strategy sets $S_1, \dots, S_n$. Each player has a payoff function π_i mapping outcomes (elements

of $S_1 \times \cdots \times S_n$) to real numbers. Recall that a potential function for such a game is defined by the following property: for every outcome $\mathbf{s} \in S_1 \times \cdots \times S_n$, every player i , and every deviation $s'_i \in S_i$.

$$\pi_i(s'_i, \mathbf{s}_{-i}) - \pi_i(s_i, \mathbf{s}_{-i}) = \Phi(s'_i, \mathbf{s}_{-i}) - \Phi(s_i, \mathbf{s}_{-i}).$$

A team game is a game in which all players have the same payoff function: $\pi_1(\mathbf{s}) = \cdots = \pi_n(\mathbf{s})$ for every outcome $\mathbf{s}$. In a dummy game, the payoff of every player i is independent of its strategy: $\pi_i(s_i, \mathbf{s}_{-i}) = \pi_i(s'_i, \mathbf{s}_{-i})$ for every $\mathbf{s}_{-i}$ and every $s_i, s'_i \in S_i$.

Prove that a game with payoffs $\pi_1, \dots, \pi_n$ is a potential game (i.e., admits a potential function) if and only if it is the sum of a team game $\pi_1^t, \dots, \pi_n^t$ and a dummy game $\pi_1^d, \dots, \pi_n^d$ (i.e., $\pi_i(\mathbf{s}) = \pi_i^t(\mathbf{s}) + \pi_i^d(\mathbf{s})$ for every i and $\mathbf{s}$).

3. Consider a combinatorial auction with n bidders and n items where each bidder i has a unit-demand valuation v_i . This means that $v_i(S) = \max_{j \in S} v_{i,j}$ for every subset S of items. We assume that $v_{i,j} > 0$ for all i, j .

In this auction, each bidder i submits one bid $b_{i,j}$ for each item j , and each item is sold separately using a second-price single-item auction. Assume that $b_{i,j} \in (0, v_{i,j})$ for all i, j . The utility of a bidder is her value for the items won, minus her total payment. For example, if bidder i has values v_{i1} and v_{i2} for two items, and wins both items when the second-highest bids are p_1 and p_2 , then her utility is $\max\{v_{i1}, v_{i2}\} - (p_1 + p_2)$. Let $G = (A, B)$ be a bipartite graph where A is the set of bidders and B is the set of items.

- (5 points) Show that every allocation π of items to bidders that maximizes the Social Welfare ($\sum_i v_{i,\pi(i)}$) induces a matching on G .
- (5 points) Show that the PoA of PNE in such a game can be at most 2.

4. (*Best response Dynamics*)

- (a) (2 points) Give an example of a game with 2 players that admits a PNE, but the best-response dynamics cycles.
- (b) This problem studies a scenario with n agents, where agent i has a positive weight $w_i > 0$. There are m identical machines. Each agent chooses a machine, and wants to minimize the load of her machine, defined as the sum of the weights of the agents who choose it. A pure Nash equilibrium in this game is an assignment of agents to machines so that no agent can unilaterally switch machines and decrease the load she experiences.

Consider the following restriction of best-response dynamics:

Algorithm 1: Maximum Weight Best-Response Dynamics

While the current outcome $\mathbf{s}$ is not a PNE:

among all agents with a beneficial deviation, let i denote an agent
with the largest weight w_i and s'_i a best response to $\mathbf{s}_{-i}$
update the outcome to $(s'_i, \mathbf{s}_{-i})$

- i. (3 points) Show that, starting from the outcome $\mathbf{s}_0$ where no agent has selected any machines (all machines have load 0), the Maximum Weight Best-Response Dynamics converges to a PNE in exactly n iterations.

- ii. (5 points) Show that, starting from any outcome $\mathbf{s}$, the Maximum Weight Best-Response Dynamics converges to a PNE in at most n iterations.
5. (*Bonus Problem*) Consider an atomic selfish routing game in which all players have the same source vertex and sink vertex (and each controls one unit of flow). Assume that edge cost functions are non-decreasing, but do not assume that they are affine. Prove that a pure-strategy Nash equilibrium can be computed in polynomial time. Be sure to discuss the issue of fractional vs. integral flows, and explain how (or if) you use the hypothesis that edge cost functions are non-decreasing.

[Hint: Recall the Rosenthal's potential function.]

6. (*Bonus Problem*) Consider n identical machines and m selfish jobs (the players). Each job j has a processing time p_j . Once jobs have chosen machines, the jobs on each machine are processed serially from shortest to longest. (You can assume that the p_j 's are distinct.) For example, if jobs with processing times 1, 3, and 5 are scheduled on a common machine, then they will complete at times 1, 4, and 9, respectively. The following questions concern the game in which players choose machines in order to minimize their completion times. The objective function as a planner is to minimize the total completion time $\sum_{j=1}^m C_j$, where C_j is the completion time job j .
- (a) Define the rank R_j of job j in a schedule as the number of jobs on j 's machine with processing time at least p_j (including j itself). For example, if jobs with processing times 1, 3, and 5 are scheduled on a common machine, then they have ranks 3, 2, and 1, respectively.
Prove that in these scheduling games, the objective function value of an outcome can also be written as $\sum_{j=1}^m p_j R_j$.
 - (b) Prove that the following algorithm produces an optimal outcome: (i) sort the jobs from largest to smallest; (ii) for $i = 1, 2, \dots, m$, assign the i^{th} job in this ordering to machine $i \bmod n$ (where machine 0 means machine n).
 - (c) Prove that for every such scheduling game, the expected objective function value of every coarse correlated equilibrium is at most twice that of an optimal outcome.