CS 580: Algorithmic Game Theory, Fall 2025
HW 1 Solutions

Instructions:

1. We will grade this assignment out of a total of 40 points.
2. You can work on any homework in groups of ($\leq$) two. Submit only one assignment per group. First submit your solutions on Gradescope and you can add your group member after submission.
3. If you discuss a problem with another group then write the names of the other group's members at the beginning of the answer for that problem.
4. Please type your solutions if possible in Latex or doc whichever is suitable, and submit on Gradescope.
5. Even if you are not able to solve a problem completely, do submit whatever you have. Partial proofs, high-level ideas, examples, and so on.
6. Except where otherwise noted, you may refer to lecture slides/notes. You cannot refer to textbooks, handouts, or research papers that have not been listed. If you do use any approved sources, make sure you **cite them appropriately**, and make sure to **write in your own words**.
7. No late assignments will be accepted.
8. By AGT book we mean the following book: Algorithmic Game Theory (edited) by Nisan, Roughgarden, Tardos and Vazirani. Its free online version is available at Prof. Vijay V. Vazirani's webpage.

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1. Consider a market with three buyers $A = \{a_1, a_2, a_3\}$ and four goods $M = \{g_1, g_2, g_3, g_4\}$. Let the budget of each buyer be \$10, and the valuation function be linear/additive where the value per unit is as given in the following table.

	g_1	g_2	g_3	g_4
a_1	4	2	2	1
a_2	2	4	1	2
a_3	1	2	2	4

   (a) (2 points) Find an allocation that is *Envy-Free* and *Pareto Optimal*.

   (b) (3 points) Find a Competitive Equilibrium allocation and prices for the above market.

   (c) (5 points) Run the DPSV'08 algorithm to find a CE prices for the above example.

Solution.

(a) We can assign g_1 to a_1, g_2 to a_2, and g_3, g_4 to a_3.

To verify envy-freeness, we can check $V_1(X_1) = 4, V_1(X_2) = 4, V_1(X_3) = 3$. So a_1 doesn't envy anyone. Similarly, $V_2(X_1) = 4, V_2(X_2) = 4, V_2(X_3) = 3$. So a_2 doesn't envy anyone. And finally, $V_3(X_1) = 1, V_3(X_2) = 2, V_3(X_3) = 6$. So a_3 doesn't envy anyone either.

To show Pareto optimality, consider that we can upper bound the social welfare as $\sum_i v_i(X_i) \leq \sum_j \max_i(v_i(g_j)) = 12$. Our allocation achieves this sum, and any Pareto dominating allocation must exceed this sum, which is impossible.

(b) We can assign $p = \{8, 10, 4, 8\}$. The corresponding demand for a_1 is $\{1, 0, 0.5, 0\}$, for a_2 is $\{0, 1, 0, 0\}$ and for a_3 is $\{0, 0, 0.5, 1\}$, which meets supply and thus is a CE.

(c) Initialize $p = \{2, 2, 1, 2\}$. We can verify the min-cut across the associated flow graph is across the p edges. We can scale p until $p = \{8, 8, 4, 8\}$, at which point we get a cross min-cut by cutting $a_1, a_3,$ and g_2.

Then we add a_1 and a_3 to A_f, and $g_1, g_3,$ and g_4 to G_f. We continue scaling the price of g_2 until $p = \{8, 10, 4, 8\}$. No new MBB edges are formed, and g_2 joins G_f. So we are done.

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2. Consider a market $\mathcal{M}$ with n agents and m divisible goods, where supply of good j is q_j and the valuation of agent i is defined by a monotonically non-decreasing concave function $V_i : \mathbb{R}^m_+ \to \mathbb{R}_+$. A *competitive equilibrium with equal income (CEEI)* of such a market is a pair (X, p) where p is a price vector $(p_1, \ldots, p_m)$, p_j is the price-per-unit of good j and $X = (X_1, \ldots, X_n)$ is an allocation of goods to agents s.t.,

- **Optimal Bundle.** For each agent i, $X_i \in \text{argmax}_{Y \geq 0: \sum_j p_j Y_j \leq 1} V_i(Y)$.
- **Demand equals Supply.** For each good j, $\sum_{i \in [n]} X_{ij} \leq q_j$, and whenever $p_j > 0$ we have $\sum_{i \in [n]} X_{ij} = q_j$.

(a) (5 points) Show that it is without loss of generality to assume that $q_j = 1$ for all goods j, that is, to assume that the supply of every good is one. Formally, come up with another market $\mathcal{M}' = ([n], [m], (q'_j)_{j \in [m]}, (V'_i)_{i \in [n]})$ such that $q'_j = 1, \ \forall j \in [m]$, and show that a CEEI of $\mathcal{M}'$ can be mapped to a CEEI of $\mathcal{M}$.

(b) (5 points) Show that when the DPSV algorithm is executed to compute a CEEI of such a market, Event 2 of the algorithm can always be computed in polynomial time. That is, the value of α for which a new MBB edge appears between some agent $i \in A_D$ and some good $j \in G_F$, and the pair (i, j), can be computed in polynomial time. Here, α is the constant by which the prices of the goods in G_D are scaled in one iteration of the algorithm. Set A_D consists of agents with MBB edges to goods in G_D before α starts increasing, and G_F is the set of goods with demand = supply, meaning if the market consisted only of the subset (A_F, G_F), then the prices of goods in G_F and their allocation to A_F satisfy the requirements of a CEEI. (See Lecture 3 slides for further details)

Solution.

(a) To define $\mathcal{M}'$, we simply need to define v'_i for all agents i. Let

$$v'_i(X'_{i1}, \dots, X'_{im}) = v_i(q_1 X'_{i1}, \dots, q_m X'_{im}).$$

Now, consider a CEEI (X', p') of $\mathcal{M}'$. We map it to a CEEI (X, p) of $\mathcal{M}$ in the following way:

- For every pair of agent i and good j, let $X_{ij} = q_j X'_{ij}$.
- For every good j, let $p_j = \frac{p'_j}{q_j}$.

Notice that since (X', p') is a CEEI of $\mathcal{M}'$, we have that for each good j,

$$\sum_i X_{ij} = \sum_i q_j X'_{ij} = q_j \sum_i X'_{ij} \le q_j$$

whenever $p_j > 0$. Also, we know that $X'_i \in \operatorname{argmax}_{Y \ge 0 : \sum_j p'_j Y_j \le 1} V_i(Y)$. It suffices to show that this is the same set of allocations in the original market, i.e. that for every agent i, $\sum_j p_j X_{ij} \le 1$, which is true, since

$$\sum_j p_j X_{ij} = \sum_j \frac{p'_j}{q_j} q_j X'_{ij} = \sum_j p'_j X'_{ij} \le 1.$$

(b) Event 2 occurs when a new MBB edge appears between a dynamic agent and a frozen good. Note that as there are no edges between the dynamic agents and the frozen goods before Event 2 occurs, the MBB value of any dynamic agent over the frozen goods is smaller than their MBB value over the dynamic goods. As the algorithm proceeds, we increase the prices of all dynamic goods by a factor of α, thereby decreasing every agent's MBB value over these goods by a factor of α.

Thus, an MBB edge will appear between a fixed dynamic agent $i \in A_D$ and some frozen good, when i's MBB value over the dynamic goods, after being lowered by α, becomes equal to their MBB value over the frozen goods. The value of α at which such an edge appears adjacent to agent i, denoted by say $\alpha(i)$, can thus be computed as follows.

The MBB value of i over the dynamic goods, after their price is increased $\alpha(i)$ times, is $\max_{j \in G_D} \frac{v_{ij}}{\alpha(i) p_j}$. The MBB value of i over the frozen goods is $\max_{j \in G_F} \frac{v_{ij}}{p_j}$.

These values become equal when $\alpha(i) = \frac{\max_{j \in G_D} \frac{v_{ij}}{p_j}}{\max_{j \in G_F} \frac{v_{ij}}{p_j}}$.

Event 2 occurs when the first such edge appears among all agents in A_D. Therefore the value of α at which Event 2 appears is,

$$\alpha = \min_{i \in A_D} \alpha(i) = \min_{i \in A_D} \frac{\max_{j \in G_D} \frac{v_{ij}}{p_j}}{\max_{j \in G_F} \frac{v_{ij}}{p_j}}.$$

This equation for α can be solved in polynomial time, therefore Event 2 can be computed in polynomial time.

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3. (*Indivisibles: Short Questions*)

(a) (2 points) Give an example with general monotone valuations where an EF1 allocation is not Prop1.

(b) (3 points) Give an example with additive valuations where the round robin algorithm achieves better social-welfare ($\sum_i V_i(A_i)$) than the envy-cycle-elimination algorithm under certain choices.

(c) (2 points) Give an example with additive valuations where an EF1+PO allocation is not EFX.

(d) (2 points) For additive valuation functions, we showed $MMS_i \leq \frac{v_i(M)}{n}$ for all agents i. Give an example with sub-additive valuation functions where this is not true, and in fact $MMS_i = v_i(M)$ for all agents i.

(e) (1 point) Prove that if an α-MMS allocation exists for an instance, then an α-MMS+PO allocation also exists.

Solution.

(a) Consider the following instance with 2 agents a_1, a_2, and 3 goods, g_1, g_2, g_3. The valuations of the agents are identical functions defined as follows. For the empty set, $v(\emptyset) = 0$, and for singleton bundles, $v(g_1) = v(g_2) = 1$, $v(g_3) = 2$. For all two-sized bundles, $v(\{g_1, g_2\}) = v(\{g_2, g_3\}) = v(\{g_1, g_3\}) = 3$ and for the grand bundle, denoted by M, $v(M) = v(\{g_1, g_2, g_3\}) = 10$.

Consider an allocation A for this instance that assigns $A_1 = \{g_1\}$ to a_1, and $A_2 = \{g_2, g_3\}$ to a_2.

For a_1, $v(A_1) \geq v(A_2 \setminus \{g_3\})$, and for a_2, $v(A_2) > v(A_1)$, and thus it is an EF1 allocation. However, for a_1 Prop1 condition is not satisfied because for each $g \in \{g_2, g_3\}$ ($\equiv M \setminus A_1$), $v(A_1 \cup \{g\}) = 3 < 5 = v(M)/2$.

(b) Consider two agents 1 and 2, and four goods a, b, c and d. The agents' valuations are $v_1(a) = 10, v_2(a) = 9, v_1(b) = v_1(c) = v_1(d) = 3, v_2(b) = v_2(c) = v_2(d) = 1$. Now, the envy-cycle elimination algorithm will allocate a to 1 and b, c and d to 2, since after a's allocation, 2 will keep envying 1. The total welfare of this allocation will be 13. However, round-robin will allocate a and c to 1 and b and d to 2, for a total welfare of 15.

(c) Consider two agents and three goods. The values of the agent 1 are $1, 0, 100$, and of 2 are $0, 1, 10$. The allocation which assigns goods a and c to agent 1, and b to agent 2 is EF1+PO, since 2 does not envy 1 after the removal of c from 1's bundle, 1 does not envy 2 at all and we cannot increase any agent's valuation without decreasing the other one's. However, 2 envies 1's allocation even after removing a from 1's bundle, hence this allocation is not EFX.

(d) Consider 2 agents i, j, and 2 items g, h. Let the valuation function of i be as follows. Their value for the empty set is 0, and the value for each singleton bundle is v. The value of both items together is also v. Then agent i can form 2 bundles of value v, hence their MMS value is v. $v(M)/2 = v/2$, hence $MMS > v(M)/2$.

(e) Any allocation that Pareto dominates an α-MMS allocation gives an equal or higher valued bundle to every agent, hence is also α-MMS. Thus, the Pareto optimal bundle among all α-MMS allocations is α-MMS+PO.

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4. (*Algorithm Design*)

(a) (5 points) Consider an instance with additive valuations where items are identically ordered. That is, there exists an ordering of items $g_1, g_2, \ldots, g_m$ such that for each agent i,

$$v_{ig_1} \geq v_{ig_2} \geq \cdots \geq v_{ig_m}.$$

Show that the *envy-cycle elimination* algorithm gives an EFX allocation when the items are considered in a particular order.

(b) (5 points) For the case with additive valuation functions, where $v_i(M) = n$ for all agents i, show that when $v_{ij} \leq \epsilon$ for all agents i and goods j, an EF1+$(1-\epsilon)$-MMS allocation exists and can be computed in polynomial time.

Solution.

(a) We let the goods be allocated in decreasing order of value. We need to argue two things:

i. That if we have a partial EFX allocation, after the envy-cycle elimination algorithm allocates one extra item, the allocation remains EFX.

ii. That when the algorithm eliminates cycles, the allocation remains EFX.

To show (i), notice that at all times, $v_i(A_j) - v_i(A_i) \leq v_i(g^*)$ where $g^*$is the good most recently added to what is currently A_j. Since bundles may have been permuted by the ency-cycle elimination step, j may not have been in possession of what is currently A_j at the time g^* was added. This does not affect the proof, however: it is sufficient to interpret A_j as "the bundle that currently belongs to j". Thus instead of saying "i did not envy j at the time", we will say "i did not envy A_j at the time". Observe that a good is allocated only to a player whom no one envies. Thus directly before g^* was added to A_j, i did not envy A_j: at that point $v_i(A_j) - v_i(A_i) \leq 0$. Therefore directly after g^* was given to j, $v_i(A_j) - v_i(A_i) \leq v_i(g^*)$. Since $v_i(A_i)$ can only have grown since then, we have $v_i(A_j) - v_i(A_i) \leq v_i(g^*)$ until a new good is added to A_j.Since the goods are allocated in decreasing order of value, the good most recently added to A_j must also be the least valuable good in A_j. Therefore at all times, $v_i(A_j) - v_i(A_i) \leq \min_{g \in A_j} v_i(g)$, and so $v_i(A_i) \geq v_i(A_j) \min_{g \in A_j} v_i(g)$. For additive valuations, this is equivalent to $v_i(A_i) \geq v_i(A_j \setminus \{g\})$ for all $g \in A_j$. Therefore the allocation at all times is EFX, so the final allocation is EFX.

To show (ii), let G be the envy graph of the allocation A before the envy-cycle elimination step. We first show that there exists another EFX allocation $A' = (A'_1, \ldots, A'_n)$ with envy graph G', where G' has strictly fewer edges than G. Let $c = (1, 2, \ldots, |c|)$ be a cycle in G. Thus $v_i(A_i) < v_i(A_{(i \mathit{\ mod\ } |c|)+1)}$ for all $i \in c$. Define a new allocation A' where $A'_i = A_{(i \mathit{\ mod\ } |c|)+1}$ for all i, and let $G' = (V', E')$ be the envy graph for A'. It is clear that A' is a permutation of A. Suppose A' is not EFX: then there exist $i, j \in N$ and $g \in A'_j$ where $v_i(A') < v_i(A'_j \setminus \{g\})$. Since A' is a permutation of A, there exists $k \in N$ where $A_k = A'_j$, so $v_i(A'_i) < v_i(A_k \setminus \{g\})$. Observe that $v_i(A'_i) > v_i(A_i)$ if $i \in c$, and $v_i(A'_i) = v_i(A_i)$ otherwise. Thus $v_i(A_i) \leq v_i(A'_i) < v_i(A_k \setminus \{g\})$, and so A is also not EFX. Therefore if A is EFX, then A' is also EFX. Note that the number of edges from $V' \setminus c$ into c is unchanged. Also, the number of edges from c into $V' \setminus c$ has decreased or stayed the same, since the utility of every player in c has strictly increased. Furthermore, for each $i \in c$, the number of players in c whom i envies has decreased by at least one. This shows that G' has strictly fewer edges than G. If G' still contains a cycle, we can apply this process again to obtain G'', G''', and so on. Since the number of edges strictly decreases each time, we can apply this process at most $|E|$ times before we obtain a envy graph without a cycle.

(b) We will show that any EF1 allocation also ensures a $(1 - \epsilon)$-MMS guarantee, thus any EF1 algorithm computes the required allocation.

As $v(M) = n$, and there are n bundles in the EF1 allocation, by the pigeonhole principle, for every agent there is at least one bundle they value at least 1. Let j denote the agent whose bundle is valued at least 1 by some agent i. Then by EF1, we have, $v_i(A_i) \geq v_i(A_j \setminus \{g\})$ for some $g \in A_j$. As the valuation functions are additive, $v_i(A_i) \geq v_i(A_j) - v_i(\{g\}) \geq 1 - \epsilon$, as $v_i(A_j) \geq 1$ and $v_i(\{g\}) \leq \epsilon$ for all g. As $MMS_i \leq v(M)/n = 1$ when the valuation functions are additive, we have $v_i(A_i) \geq (1 - \epsilon) MMS_i$.