More Goods Are All You Need

An asymptotically improved EFX approximation

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Preliminaries

Background

- Agents $\{1, \ldots, n\}$ and a set X of m indivisible goods $g_1, \ldots, g_m$
- Each agent i has a valuation function $v_i : \mathcal{P}(X) \to \mathbb{R}_+$
  - Common restrictions: submodular, additive, ...
- An allocation $(X_1, \ldots, X_n)$ gives goods in X_i to each agent i, satisfying $X_i \cap X_j = \emptyset$ for all i, j, and $\cup X_i \subseteq X$
  - Unallocated goods are said to go to “charity” [4]
- An *instance* $\langle n, \{v_1, \ldots, v_n\}, X \rangle$ consists of agents, valuation functions, and the pool of goods

Characterizing Valuation Functions

We focus our attention on *additive* valuations, which satisfy $v_i(X') = \sum_{x \in X'} v_i(x)$ for all $X' \subseteq X$

Definition

For valuation v and set X, denote $x_i^{(v)}$ to be the item of rank i when ordering X by v, ascending

(σ, d)-differing Valuations

Definition (d-difference bounded)

$v : X \to \mathbb{R}_+$ is d-difference bounded if $v(x_i^{(v)}) + d \geq v(x_{i+1}^{(v)})$ for all $i < |X|$, and $v(x_1^{(v)}) \leq d$. An allocation instance is d-difference bounded, if for each agent i, there is some d_i so that v_i is d_i-difference bounded and $d = \max_i d_i$.

Definition (σ-difference required)

$v : X \to \mathbb{R}_+$ is σ-difference required if $v(x_{i+1}^{(v)}) - v(x_i^{(v)}) \geq \sigma$ for all $i < |X|$, and $v(x_1^{(v)}) \geq \sigma$. An allocation instance is σ-difference required, if for each agent i, there is some σ_i so that v_i is σ_i-difference required and $\sigma = \min_i \sigma_i$.

ρ-uniform Instances

Definition (ρ-uniform)

An allocation instance is (σ, d)-differing if it is σ-difference required and d-difference bounded. We call $\rho = \frac{\sigma}{d}$ the *uniformity* of the instance.

Intuitively, ρ is a measure of how evenly spread item valuations are: $\rho = 1$ occurs when items are ranked (up to a constant factor) $1, 2, \ldots, m$, $\rho = 0$ occurs when there are two items with the same value (a degenerate instance).

Theorem ([2])

If an EFX allocation always exists for n agents with non-degenerate additive valuation functions, then an EFX allocation always exists for n agents with any additive valuation functions.

Corollary

For any allocation instance, we can assume wlog $\rho > 0$.

Envy

Definition

Agent i *envies* agent j if $v_i(X_i) < v_i(X_j)$

Definition

Agent i *strongly* envies agent j if $v_i(X_i) < v_i(X_j \setminus h) \ \ \forall h \in X_j$

Definition ([1])

An allocation $(X_1, \dots)$ is *EFX* if no agent strongly envies another

EFX Approximations

Definition

An allocation is α-EFX if $v_i(X_i) \geq \alpha v_i(X_j \setminus h)$ for all $h \in X_j$.

Theorem ([6])

For n agents with additive valuations, there exists a 1/2-EFX allocation.

Theorem ([3])

For n agents with additive valuations, there exists a 0.618-EFX allocation.

Our Results

An asymptotically improved EFX approximation factor

Theorem

For a ρ-uniform valuation instance with $m \geq \frac{4n^2}{\rho}$, there exists a $\left(1 - \frac{4n^3}{\rho m^2}\right)$-EFX allocation, and it can be computed in polytime.

Note: Yesterday we improved the bound to $\left(1 - \frac{n^3}{\rho m^2}\right)$ and removed the $m \geq \frac{4n^2}{\rho}$ requirement, but still need to review it more thoroughly.

For $n = 4$ agents and $\rho = 0.01$, this is better than 0.618-EFX when $m \geq 130$.

The Envy Graph

Definition (Envy Graph)

Given a partial allocation $(X_1, \ldots, X_n)$, define the envy graph $G^E = ([n], \{i \to j : v_i(X_i) < v_i(X_j)\}$.

Lemma ([5])

If cycles exist in G^E, bundles may be swapped so that $G^{E'}$ is acyclic.

Topological Assignment

Assume $m = cn$ for some c, and start with all items unallocated. Let G^E be the envy graph. Play c rounds as follows:

1. Compute a topological ordering of G^E
2. In this order, allow agents to pick their most valued available item
3. If envy cycles exist in G^E, eliminate them following the procedure described in lecture [5]

Bounding Envy

Lemma (Envy from Topological Assignment)

After each round, $v_i(X_j) - v_i(X_i) \leq dn$*.*

Proof.

By induction. First, consider the first round. Agent 1 picks their favorite item. Then, agent 2 picks their favorite item, or if agent 1 took their favorite item, then agent 2 gets their 2nd favorite item. This continues, up until agent n, who gets at worst their (n)th favorite item. Since the valuations are d-difference bounded, the difference in agent n's value of his favorite and nth favorite items is at most $d(n-1)$. In the worst case, agent n gets her nth favorite item and another agent j got her favorite item, so $v_n(X_j) - v_n(X_n) \leq d(n-1) \leq dn$. No agent gets worse than their nth favorite item in the first round, so the claim holds. □

Bounding Envy Cont.

Proof.

Now suppose we have completed an arbitrary round k. Consider arbitrary i and j. Let g_i and g_j be the goods picked by agents i and j this round, respectively. Let X_i and X_j be the bundles of i and j before entering this round. By induction, we know that $v_i(X_j) - v_i(X_i) \leq dn$. We want to prove this inequality for $v_i(X_j \cup g_j) - v_i(X_i \cup g_i)$ as well. We have two cases:

1. Agent i picked before agent j. Then we know that $v_i(g_i) \geq v_i(g_j) \implies v_i(g_j) - v_i(g_i) \leq 0$, since agent i picked before agent j and chose item g_i over item g_j. Thus, after the kth round, agent i has bundle $X_i \cup g_i$ and agent j has bundle $X_j \cup g_j$. Then

$$v_i(X_j \cup g_j) - v_i(X_i \cup g_i) = v_i(X_j) - v_i(X_i) + v_i(g_j) - v_i(g_i) \leq v_i(X_j) - v_i(X_i) \leq dn$$

□

Bounding Envy Cont.

Proof.

2. Agent j picked before agent i. But, since we picked in *topological order*, agent i did not envy agent j's bundle before the start of this round. Thus, $v_i(X_j) - v_i(X_i) \leq 0$. Now, we notice that the maximum amount agent i prefers g_j over g_i is $d(n-1)$ (similar reasoning as in the base case). Thus, $v_i(g_j) - v_i(g_i) \leq dn$. This gives us

$$v_i(X_j \cup g_j) - v_i(X_i \cup g_i) = v_i(X_j) - v_i(X_i) + v_i(g_j) - v_i(g_i) \leq v_i(g_j) - v_i(g_i) \leq dn$$

Now, we observe that envy cycle elimination preserves these pairwise inequalities: each agent in the cycle gets a bundle they previously envied, so their value of their bundles strictly increases. Other edges are shifted, but the inequalites remain since bundles are not modified. □

Approximating EFX

Theorem

For a ρ-uniform valuation instance with $m \geq \frac{4n^2}{\rho}$, there exists a $\left(1 - \frac{4n^3}{\rho m^2}\right)$-EFX allocation, and it can be computed in polytime.

Proof.

Consider agents i and j. If $v_i(X_i) \geq v_i(X_j)$, we are done, so assume otherwise. By the previous lemma,

$$v_i(X_i) \geq v_i(X_j) - dn$$

Let $e_{ij} = v_i(X_j) - \min_{x \in X_j} v_i(X)$. We can rewrite the above in terms of e_{ij} as follows:

$$v_i(X_i) \geq e_{ij} \left(1 - \frac{dn - \min_{x \in X_j} v_i(x)}{v_i(X_j) - \min_{x \in X_j} v_i(x)}\right)$$

□

Approximating EFX Cont.

Proof.

Since valuations are (σ, d)-differing, we have

$$v_i(X_i) \geq e_{ij}\left(1 - \frac{dn}{\sigma\frac{\frac{m}{n}\left(\frac{m}{n}+1\right)}{2} - \left(dm - \sigma\frac{m}{n}\right)}\right) \geq e_{ij}\left(1 - \frac{dn}{\sigma\frac{m^2}{2n^2} - dm}\right)$$

We can factor out a d:

$$v_i(X_i) \geq e_{ij}\left(1 - \frac{n}{\frac{m^2\rho}{2n^2} - m}\right)$$

And from here, we may simplify algebraically:

$$v_i(X_i) \geq e_{ij}\left(1 - \frac{n}{m\left(\frac{m\rho}{2n^2} - 1\right)}\right) \geq e_{ij}\left(1 - \frac{n}{\frac{m^2\rho}{4n^2}}\right) = e_{ij}\left(1 - \frac{4n^3}{\rho m^2}\right)$$

□

When $m \neq cn$?

Since the instance is (σ, d)-differing, we can add at most $n-1$ items each agent values at 0, and will not affect our proofs: instance is still d-difference bounded, and consecutive items still have spacing at least σ. The previous proof will have $(m/n-1)$ instead of $(m/n+1)$, but this doesn't change much.

Future Work

Our Failed Ideas

- Instead of topological order, order agents by valuation of their bundle.
  - Seems to have good MMS guarantees for $\rho = 1$, but does not work well for smaller ρ
  - What might work: sort agents by fraction of total value obtained (i.e. $v_i(X_i)/v_i(X)$)
- Chip firing games: The following procedure seems to always converge to an EFX allocation regardless of valuations:
  1. While there is strong envy from i to j, donate an item from j to i
- Conjecture: EFX existence could be demonstrated using combinatorial techniques similar to chip firing stability results
- Conjecture: our result holds for bounded marginals
- Our EFX approximation could probably be improved by being more flexible in assigning items - post-topological assignment item movement might be low-hanging fruit

References i

I. Caragiannis, N. Gravin, and X. Huang.
Envy-freeness up to any item with high nash welfare: The virtue of donating items.
CoRR, abs/1902.04319, 2019.

B. R. Chaudhury, J. Garg, and K. Mehlhorn.
EFX exists for three agents.
CoRR, abs/2002.05119, 2020.

B. R. Chaudhury, J. Garg, K. Mehlhorn, R. Mehta, and P. Misra.
Improving EFX guarantees through rainbow cycle number.
CoRR, abs/2103.01628, 2021.

B. R. Chaudhury, T. Kavitha, K. Mehlhorn, and A. Sgouritsa.
A little charity guarantees almost envy-freeness.
CoRR, abs/1907.04596, 2019.

References ii

R. J. Lipton, E. Markakis, E. Mossel, and A. Saberi.
On approximately fair allocations of indivisible goods.
In *Proceedings of the 5th ACM Conference on Electronic Commerce*, EC '04, page 125–131, New York, NY, USA, 2004. Association for Computing Machinery.

B. Plaut and T. Roughgarden.
Almost envy-freeness with general valuations.
CoRR, abs/1707.04769, 2017.