Stable Matching, House allocation, Kidney exchange

Thursday, September 9, 2021 12:56 PM

☆ Stable Matching ☆

Applications : Assigning doctors to hospitals
students to schools

★ Problem :

Agents

(men) U — $(v_1 \geq v_3 \geq v_5 \dots)$ u_1, u_2, $\vdots$, u_n

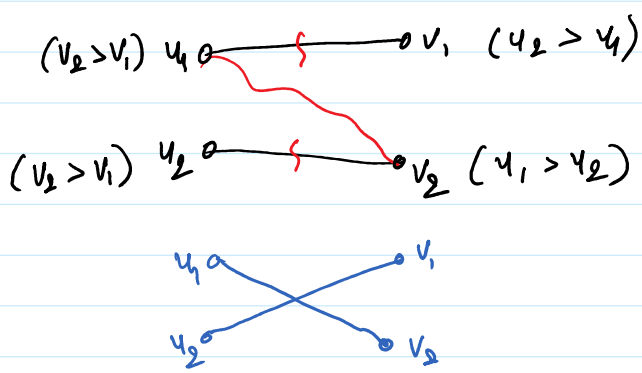
V (women) — v_1 $(u_3 \geq u_5 \geq u_4 \dots)$, v_2, $\vdots$, v_n

↑ Complete

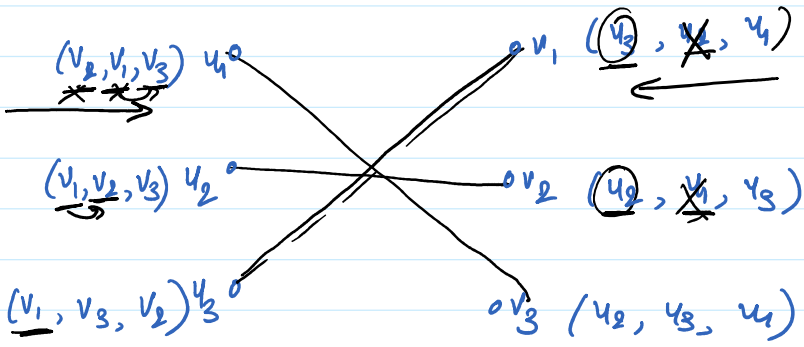
★ Goal : Find a "stable" perfect matching.

- Blocking pair

A matching is stable if no blocking pair



☆ Gale-Shapley Algo (Differed Acceptance Algo)



Running-time : $O(n^2)$
Why o/p perfect matching ?

Thm : GS o/p stable perfect matching.

Pf : M : GS o/p

$(u,v) \in M$, $u \in U$, $v \in V$

Case I : u never proposed to v.

suppose $(u, v') \in M$

then $v' >_u v$

Case II : u proposes to v but was rejected in favour of u'

then $u' >_v u$

Finally, if $(u'', v) \in M$ then $u'' >_v u' >_v u$

Either $(v' > v)$ u o———o v'

or

v'' o———o v $(v'' > u)$

★ Thm : It o/p men optimal stable matching.
& worst stable matching for women.

"Men opt stable matching" :

$$\text{best-SM}(u) = \underset{v}{\operatorname{argmax}} \left\{ v \,/\, (u,v) \in \text{some stable matching} \right\}$$

$$MOSM \ni (u, \text{best-SM}(u)) \quad \forall u.$$

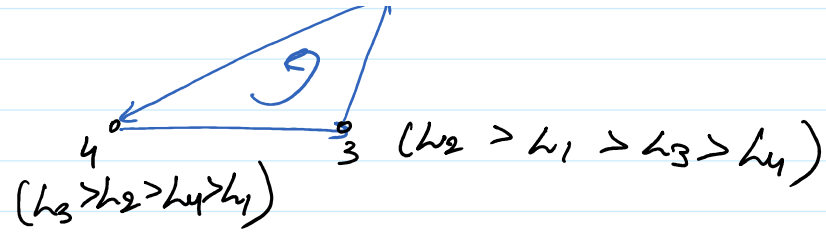
★ House Allocation

A = set of n agent, each with a house. Agent i owns House h_i
$i \in A$ has a complete preference over $h_1, \dots h_n$

$(h_2 > h_3 > h_1 > h_4)$ 1 o

2 o $(h_4 > h_1 > h_3 > h_2)$

3



★ Top Trading Cycle (TTC):

① $A = \{1, \dots n\}$, $H = \{1 \dots n\}$

② while $A \neq \emptyset$

(2.1) each $i \in A$ points to her most preferred house in H, $\rightarrow G$

(2.2) $\mathcal{C}$: All cycles in the graph G. (No sink in $G \Rightarrow$ Not a DAG $\Rightarrow \exists$ a cycle)

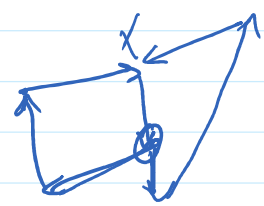
(2.3) Exchange houses along every cycle in $\mathcal{C}$. $A = A \setminus$ agents in $\mathcal{C}$, $H = H \setminus$ houses of agents in $\mathcal{C}$

OBS 1: A node can be part of at most one cycle in G.

Claim 1. Every agent only improves.

i $\quad h_i \succ_i h_k$

G:



★ Strongest Notion of truthfulness: Dominant Strategy incentive compatible (DSIC)

An agent has "no" incentive to lie no matter what others do.

Thm. TTC is DSIC.

Pf. N_j = set of agent who gets assigned in Round j.

OBS 2. No agent in N_1 has any incentive to lie.

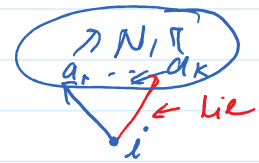
OBS 3. By lying an agent can only change her outgoing edges in G in any round, not the incoming edges.

OBS 4: Let $i \in N_2$ & $h_1^i \succ_i h_2^i \succ_i \dots h_n^i$. Suppose $h_1^i, \dots, h_k^i$ are taken in Round 1 by agents $a_1, \dots a_k \in N_1$, & i gets $h_{(k+1)}^i$

$\Rightarrow a_1$ to a_k are not pointing to i in Round 1.

And no matter what preference order agent i reports she cannot change this fact.

$\Rightarrow$ i has no way to get any of $h_1^i, \dots, h_k^i$.



$\Rightarrow$ Since, Out of $H \setminus (\text{Houses of } N_1)$, i is getting the most preferred house, she can not lie & get better.

OBS 5: Let $i \in N_j$ & $h_1^i >_i h_2^i >_i \dots >_i h_n^i$

If the first available house in this order after $1, \dots, (j-1)$ rounds is $h_{(k+1)}^i$ then no matter what i reports she cannot get any of $h_1^i, \dots, h_k^i$

The argument is the same as above. $h_1^i, \dots h_k^i$ are assigned to some k agents in $N_1 \cup \dots \cup N_{j-1}$ $\Rightarrow$ None of them point to i in first $(j-1)$ rounds $\Rightarrow$ No matter what i reports she can not get any of $h_1^i, \dots, h_k^i$. ∎

NOTE: The above proof also shows that if we run the algorithm not as a blackbox but online, actually asking agents to point to their most preferred house among the available ones, still it is DSIC.

WHY?