Revenue Maximization

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→ welfare maximization → v/s DSIC + revenue maximization

→ 1 item 1 bidder

mechanism → reserve price

$$\text{revenue} = \begin{cases} r & \to \text{ if } v \geq r \\ 0 & \text{o.w.} \end{cases}$$

$$\text{welfare} = \begin{cases} v - r & \to \text{ if } v \geq r \quad (v) \\ 0 & \to v < r \end{cases}$$

⇒ Expected revenue.

Model

ⓐ single parameter environment.

ⓑ n bidders. Seller does not know private vals $v_1, \dots, v_n$

knows for each i, $v_i \sim F_i$, f_i ; → $[0, v^{max}]$.

ⓒ DSIC → $b_i = v_i$

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examples:

① 1 bidder, 1 item → reserve price.

↓

$v_i \sim [0,1]$ – uniform distribution

$$\mathbb{E}[\text{revenue}] = r \cdot \Pr[v \geq r] + \cancel{0 \Pr[v < r]}$$

$$= r[1 - F(r)]$$

$$= r[1 - r]$$

$$= r - r^2.$$

$$1 - 2r = 0$$

$$\underline{r = 1/2}$$

② 1 item, 2 bidders → $F_1, F_2 \sim [0,1]$ uniform distribution

→ v_1, v_2 $[v_1 < v_2]$

$$\mathbb{E}[\text{revenue}] = \int^1 \int^{v_2} v_1 \, dv_1 \, dv_2.$$

$\to \quad v_1, v_2$ $\quad (v_1 < v_2)$

$$\mathbb{E}_{v_1 < v_2}[\text{revenue}] = \int_0^1 \int_0^{v_2} v_1 \, dv_1 \, dv_2$$

$$= \int_0^1 \frac{v_1^2}{2}\Big|_0^{v_2} dv_2 = \int_0^1 \frac{v_2^2}{2} dv_2 = \frac{v_2^3}{6}\Big|_0^1$$

$$= \frac{1}{6}$$

$$\mathbb{E}[\text{revenue}] = 2 \cdot \frac{1}{6} = \underline{\underline{\frac{1}{3}}}$$

(3) 1 item, 2 bidders $\quad r = 1/2$. $\quad$ revenue $= \max\{\frac{1}{2}, 2^{nd} \text{ highest bid}\}$

$\to \quad \mathbb{E}[\text{revenue}] = 0 \quad$ w.p. $\frac{1}{4} \to v_1, v_2 \le \frac{1}{2}$.

$\frac{1}{2} \quad$ w.p. $\frac{1}{2} \to v_1 \le \frac{1}{2} \le v_2$

$v_2 \le \frac{1}{2} \le v_1$

$v_2 \quad$ w.p. $\frac{1}{4} \to$ o.w

$$\frac{1}{2} \cdot \frac{1}{2} + \frac{2}{3} \cdot \frac{1}{4} = \left(1 + \frac{2}{3}\right)\frac{1}{4} = \frac{5}{12} > \frac{1}{3}.$$

Goal : auction that F_is, f_is, $[0, v^{max}]$, n.
maximizes $\mathbb{E}[\text{revenue}]$ (DSIC)

$\to \quad \mathbb{E}_v[p(v)]$

$\to \quad \mathbb{E}_v\left[\sum_{i=1}^{n} p_i(v)\right]$

$$= \mathbb{E}_{-v_i} \mathbb{E}_{v_i}\left[\sum_{i=1}^{n} p_i(v)\right]$$

$$= \underline{\mathbb{E}_{-v_i} \sum_{i=1}^{n}} \mathbb{E}_{v_i}[p_i(v)]$$

$$\mathbb{E}_{v_i}[p_i(v)] = \int_0^{v_{max}} p_i(v) f_i(v_i) dv_i$$

$$= \int_0^{v_{max}} \left(\int_0^{v_i} z \cdot x_i'(z, v_{-i}) dz \right) f_i(v_i) dv_i$$

$\int^{v_{max}} / \int^{v_{max}}$

$$= \int_0^{v_{max}} \underbrace{\left(\int_z^{v_{max}} f_i(v_i)\, dv_i \right)}\, z \cdot x_i'(z, v_{-i})\, dz .$$

$$= \int_0^{v_{max}} \underbrace{(1 - F_i(z)) \cdot z}_{u} \cdot \underbrace{x_i'(z, v_{-i})}_{v}\, dz .$$

$$= \cancel{(1 - F_i(z)) \cdot z \cdot (x_i(z, v_{-i})) \Big|_0^{v_{max}}} - \int_0^{v_{max}} \underbrace{\left((1 - F_i(z)) - z f_i(z) \right)} x_i(z, v_{-i})\, dz$$

$$= \int_0^{v_{max}} \underbrace{\left(z - \frac{1 - F_i(z)}{f_i(z)} \right)}_{\phi_i(z)} x_i(z, v_{-i})\, f_i(z)\, dz .$$

'virtual welfare of i'

$$(v_i \cdot x(v))$$

$$\mathbb{E}_{v_i}[p_i(v)] = \mathbb{E}_{v_i \sim F_i}\left[\phi_i(v_i) \cdot x(v_i) \right]$$

$$\underbrace{v_i - \frac{1 - F_i(v_i)}{f_i(v_i)}}$$

$$\mathbb{E}_v[p(v)] = \sum_{i=1}^{n} \mathbb{E}_v[\phi_i(v_i) \cdot x_i(v)] = \mathbb{E}_v\left[\sum_{i=1}^{n} \phi_i(v_i) \cdot x_i(v) \right]$$

EXPECTED REVENUE = EXPECTED VIRTUAL WELFARE

- 1 item, n bidders, $F_i = F \ \forall i$.

$\rightarrow$ $$\mathbb{E}_v\left[\sum_{i=1}^{n} \phi_i(v_i)\, x_i(v) \right]$$

$v_i \sim [0,1]$

$$\phi(v_i) = v_i - \frac{1 - v_i}{1} = 2v_i - 1$$

$$i^* \in \arg\max \phi(v_i)$$

$$\phi(v_{i^*}) \geq 0 \rightarrow \underbrace{v_{i^*} \geq \phi^{-1}(0)}$$

Rule: F is "regular" $\rightarrow$ $\phi(v)$ is monotonically $\uparrow$ with v.

$F_1 \ldots F_n$

$v_1 \ldots v_n$

$$\phi_1(v_1) \geq \phi_2(v_2) \geq \ldots \geq \phi_n(v_n).$$

Prior independent Auctions

Bulow-Klemperer thm:

$$\mathbb{E}_v[VA^{n+1}(v)] \geq \mathbb{E}_v[OPT_F^n(v)]$$

$\rightarrow$ $A:(n+1)$

- runs OPT for n bidders $\rightarrow$ give item to i if OPT allocated to i
- if item was unallocated, assign to bidder $(n+1)$ for free.

$$\mathbb{E}\left[\begin{matrix}\text{Revenue of VA} \\ \text{over } n+1 \text{ bidders}\end{matrix}\right] \geq \mathbb{E}[\text{revenue from } A] = \mathbb{E}[OPT_F^n(v)]$$

(VA = OPT when item has to be allocated