Rewrite Theories in General

It is clear that rewriting logic has the underlying equational logic as a parameter: the more general the equational logic, the more general the resulting rewite theories. For example, we have seen that for order-sorted equational logic rules can have the general form,

$$l : t \longrightarrow t' \;\Leftarrow\; (\bigwedge_i u_i = u'_i) \wedge (\bigwedge_j w_j \longrightarrow w'_j).$$

It has also become increasingly clear that frozen operators, that restrict the rewrites allowed below them, are also very useful in practice.

Rewrite Theories in General (II)

We can illustrate frozen operators with the following nondeterministic choice example (in Maude syntax):

```
mod CHOICE is
  protecting INT .
  sorts Elt MSet .
  subsorts Elt < MSet .
  ops a b c d e f g : -> Elt .
  op __ : MSet MSet -> MSet [assoc comm] .
  op card : MSet -> Int [frozen] .
  eq card(X:Elt) = 1 .
  eq card(X:Elt M:MSet) = 1 + card(M:MSet) .
  rl [choice] : X:MSet Y:MSet => Y:MSet .
endm
```

Rewrite Theories in General (III)

It does not make much sense to rewrite below the cardinality function `card`, because then the multiset whose cardinality we wish to determine becomes a moving target.

If `card` had not been declared `frozen`, then the rewrites, `a b c` $\longrightarrow$ `b c` $\longrightarrow$ `c` would induce rewrites, $3 \longrightarrow 2 \longrightarrow 1$, which seems bizarre.

The point is that we think of the kind `[MSet]` as the state kind in this example, whereas `[Int]` is the data kind. By declaring `card` frozen, we restrict rewrites to the state kind, where they belong.

Rewrite Theories in General (IV)

This leads to the following general definition of a rewrite theory on order-sorted equational logic:

A rewrite theory is a 4-tuple, $\mathcal{R} = (\Sigma, E, \phi, R)$, where:

- (Σ, E) is a kind-complete order-sorted equational theory, with, say, kinds K, sorts S, and operations Σ
- $\phi : \Sigma \longrightarrow \mathcal{P}_{fin}(\mathbb{N})$ is a $K^* \times K$-indexed family of functions assigning to each $f : k_1 \ldots k_n \longrightarrow k$ in Σ the finite set $\phi(f) \subseteq \{1, \ldots, n\}$ of its frozen argument positions
- R is a set of (universally quantified) labeled conditional rewrite rules of the form (with t, t' and the w_k, w'_k pairs of terms of same kind)

$$l : t \longrightarrow t' \;\Leftarrow\; (\bigwedge_i u_i = u'_i) \wedge (\bigwedge_j w_j \longrightarrow w'_j).$$

Rewrite Theories in General (V)

Given a rewrite theory $\mathcal{R} = (\Sigma, E, \phi, R)$, and given a Σ-term $t \in T_\Sigma(X)$, we call a variable $x \in vars(t)$ frozen in t iff there is a nonvariable position $\alpha \in \mathbb{N}^*$ such that $t/\alpha = f(u_1, \ldots, u_i, \ldots, u_n)$, with $i \in \phi(f)$, and $x \in vars(u_i)$. Otherwise, we call $x \in X$ unfrozen.

Similarly, given Σ-terms $t, t' \in T_\Sigma(X)$, we call a variable $x \in X$ unfrozen in t and t' iff it is unfrozen in both t and t'.

Rewriting Logic in General

Given a rewrite theory $\mathcal{R} = (\Sigma, E, \phi, R)$, the sentences that it proves are universally quantified rewrites of the form, $(\forall X)\ t \longrightarrow t'$, with $t, t' \in T_{\Sigma,E}(X)_k$, for some kind k, which are obtained by finite application of the following rules of deduction:

- Reflexivity. For each $t \in T_\Sigma(X)$, $\quad \dfrac{}{(\forall X)\ t \longrightarrow t}$

- Equality. $$\frac{(\forall X)\ u \longrightarrow v \quad E \vdash (\forall X) u = u' \quad E \vdash (\forall X) v = v'}{(\forall X)\ u' \longrightarrow v'}$$

- Congruence. For each $f : k_1 \ldots k_n \longrightarrow k$ in Σ, with $\{1, \ldots, n\} - \phi(f) = \{j_1, \ldots, j_m\}$, with $t_i \in T_\Sigma(X)_{k_i}$, $1 \leq i \leq n$, and with $t'_{j_l} \in T_\Sigma(X)_{k_{j_l}}$, $1 \leq l \leq m$,

$$\frac{(\forall X)\ t_{j_1} \longrightarrow t'_{j_1} \quad \ldots \quad (\forall X)\ t_{j_m} \longrightarrow t'_{j_m}}{(\forall X)\ f(t_1, \ldots, t_{j_1}, \ldots, t_{j_m}, \ldots, t_n) \longrightarrow f(t_1, \ldots, t'_{j_1}, \ldots, t'_{j_m}, \ldots, t_n)}$$

- Replacement. For each finite substitution $\theta : X \longrightarrow T_\Sigma(Y)$, with, say, $X = \{x_1, \ldots, x_n\}$, and $\theta(x_l) = p_l$, $1 \leq l \leq n$, and for each rule in R of the form,

$$l : (\forall X)\ t \longrightarrow t' \ \Leftarrow\ (\bigwedge_i u_i = u'_i) \wedge (\bigwedge_j w_j \longrightarrow w'_j)$$

with $Z = \{x_{j_1}, \ldots, x_{j_m}\}$, the set of unfrozen variables in t and t', then,

$$(\bigwedge_r (\forall Y)\ p_{j_r} \longrightarrow p'_{j_r})$$

$$\frac{(\bigwedge_i (\forall Y)\ \theta(u_i) = \theta(u'_i)) \quad \wedge \quad (\bigwedge_j (\forall Y)\ \theta(w_j) \longrightarrow \theta(w'_j))}{(\forall Y)\ \theta(t) \longrightarrow \theta'(t')}$$

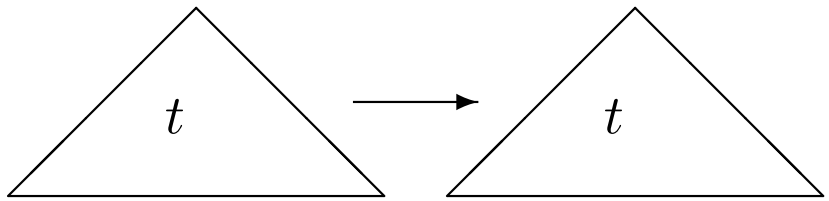
where for $x \in X - Z$, $\theta'(x) = \theta(x)$, and for $x_{j_r} \in Z$, $\theta'(x_{j_r}) = p'_{j_r}$, $1 \leq r \leq m$.

- Transitivity

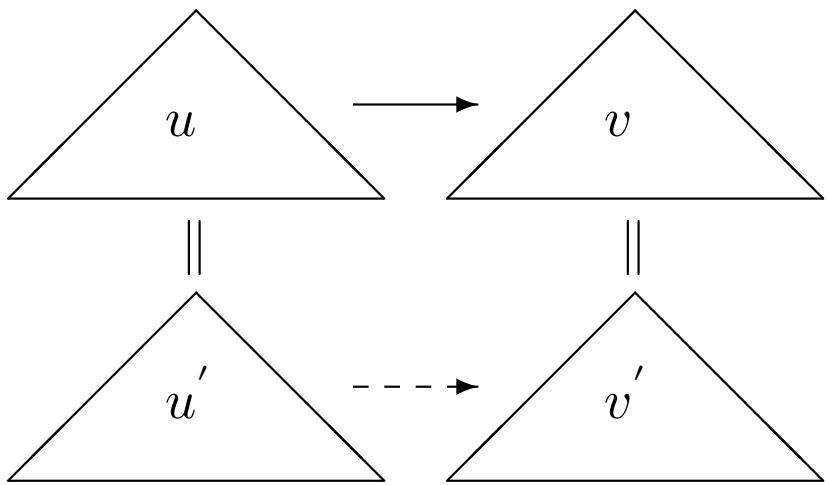
$$\frac{(\forall X)\ t_1 \longrightarrow t_2 \qquad (\forall X)\ t_2 \longrightarrow t_3}{(\forall X)\ t_1 \longrightarrow t_3}$$

Rewriting Logic in Pictures

Reflexivity

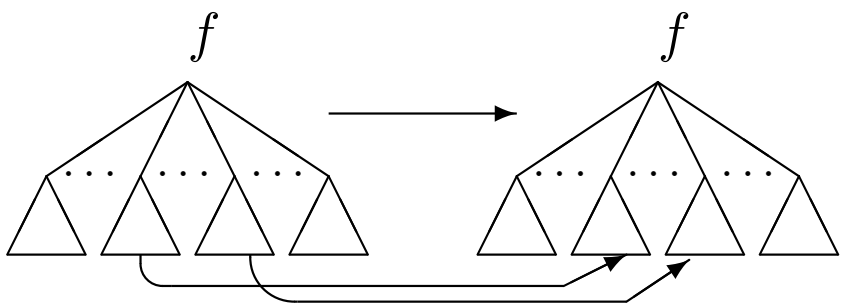


Equality

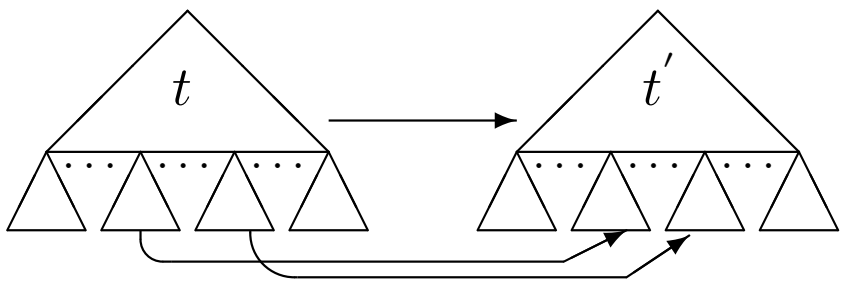


Rewriting Logic in Pictures (II)

Congruence



Replacement



Rewriting Logic in Pictures (III)

Transitivity