QUANTUM STATE TOMOGRAPHY

Suppose you have an unknown quantum state (obtained, e.g., as a gift from a stranger who teleported it to you)

Tomography is the following task:

Given (sufficiently) many copies of an unknown quantum state ρ, learn the density matrix of ρ.
"approximately"

Imagine you have a button that outputs ρ each time you press it.

FIRST EXAMPLE

Learning a single qubit pure state $|\psi\rangle = \begin{bmatrix} \alpha \\ \beta \end{bmatrix}$

Its density matrix $\rho = |\psi\rangle\langle\psi| = \begin{bmatrix} |\alpha|^2 & \alpha\beta^* \\ \beta\alpha^* & |\beta|^2 \end{bmatrix}$

Measure in standard basis.

Yields "0" w.p. $|\alpha|^2$, "1" w.p. $|\beta|^2$

IN-CLASS : are we done ?

Learning single-qubit mixed state $\rho = \begin{bmatrix} a & b \\ b^* & c \end{bmatrix}$

because, Hermitian

Measure in standard basis

Yields "0" w.p. $\langle 0 | \rho | 0 \rangle = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} a & b \\ b^* & c \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = a$

Yields "1" w.p. $\langle 1 | \rho | 1 \rangle = c$

How should we learn b?

Measure in Hadamard basis?

Yields "+" w.p. $\langle + | \rho | + \rangle = \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix} \begin{bmatrix} a & b \\ b^* & c \end{bmatrix} \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix}$

$$= \frac{1}{\sqrt{2}} \begin{bmatrix} a + b^* & b + c \end{bmatrix} \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix}$$

$$= \frac{1}{2} \left(a + b + b^* + c \right)$$

This gives us an estimate of $(b + b^*) = 2\,\mathrm{Re}(b)$

Measure in $|+i\rangle, |-i\rangle$ basis

Yields "+i" w.p. $\langle +i|\rho|+i\rangle = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{-i}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} a & b \\ b^* & c \end{bmatrix} \begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{i}{\sqrt{2}} \end{pmatrix}$

$$= \frac{1}{\sqrt{2}} \begin{bmatrix} a - ib^* & b - ic \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{i}{\sqrt{2}} \end{bmatrix} = \frac{1}{2}\left(a - ib^* + ib - i^2 c\right)$$

$= \frac{1}{2}\left(a + c + i(b - b^*)\right)$. This gives us Complex(b).

What about bigger matrices ?

D - dimensional Hilbert Space

Goal : Learn $D \times D$ density matrix [$D = 2^n$, n qubit state]

$$\rho = \begin{bmatrix} \rho_{11} & \rho_{12} & \cdots & \rho_{1n} \\ \vdots & \rho_{22} & & \vdots \\ \vdots & \vdots & \ddots & \vdots \\ \rho_{n1} & \rho_{n2} & \cdots & \rho_{nn} \end{bmatrix}$$

Measure ρ in standard basis

Pr "1" $= \rho_{11}$ — Pr "2" $= \rho_{22}$ ..... Pr "n" $= \rho_{nn}$

So this gives us the diagonal entries

How should we learn off-diagonal entries?

Rotate the state to $\rho' = \tilde{H}\rho$ where

$$\rho' = \begin{bmatrix} 1 & 1 & & & & 0 \\ 1 & -1 & & & & \\ & & \ddots & & & \\ & & & \ddots & & \\ & 0 & & & 1 & 1 \\ & & & & 1 & -1 \end{bmatrix} \begin{bmatrix} \rho_{11} & \rho_{12} & \cdots & \rho_{1n} \\ \vdots & \rho_{22} & & \vdots \\ \vdots & \vdots & \ddots & \vdots \\ \rho_{n1} & \rho_{n2} & \cdots & \rho_{nn} \end{bmatrix}$$

$$= \begin{bmatrix} \rho_{11}+\rho_{21} & \rho_{12}+\rho_{22} & & & \\ \rho_{11}-\rho_{21} & \rho_{12}-\rho_{22} & & & \\ & & \ddots & & \\ & & & & \end{bmatrix}$$

standard basis msmt gives $\rho_{11}+\rho_{21} \rightarrow$ gives ρ_{21}

also gives $\rho_{12}-\rho_{22} \rightarrow$ gives ρ_{12}

Use this to compute the 2×2 blocks shaded in pink above

What about the remaining blocks?

Permute rows of the matrix you used, for example, swith the 2^{nd} & 3^{rd} row of $\tilde{H}$

Then $\rho' = \begin{bmatrix} \rho_{11}+\rho_{31} & \rho_{13}+\rho_{33} & & \\ \rho_{11}-\rho_{31} & \rho_{13}-\rho_{33} & & \\ & & \ddots & \\ & & & \ddots \end{bmatrix}$

Now apply standard basis measurements and compute as above.

Each term we approximate has error $\pm \delta$

(This requires $O\left(\frac{1}{\delta^2}\right)$ samples per term)

Number of terms $= D$.

So, total copies $= O\left(\frac{D}{\delta^2}\right)$

TESTING QUANTUM ADVANTAGE

Race to build quantum computers

Today : Quantum computers with 50+ (very) noisy qubits

Q : How should we verify that a quantum device is in fact doing something classical computers cannot?

1) Use a near-term, noisy device to generate some values that are provably (modulo some reasonable assumptions) hard for classical computers.

2) Verify that this task is done correctly?

FACTORING?

50-qubit computers cannot factor very large numbers, classical computers can also factor these numbers.

RANDOM CIRCUIT SAMPLING

n qubit circuit (say ~50)

m gates (several 100)

1. Pick random quantum circuit C with m gates from a universal gate set on n qubits.

2. Run $C|0^n\rangle$ and measure output in standard basis. Call the resulting string $x \in \{0,1\}^n$.

3. Repeat the above T times (T = 1 million roughly) to obtain $(x_1, x_2, \ldots, x_T)$

$$\forall x \in \{0,1\}^n, \quad \Pr[x] = \left| \langle x | C | 0^n \rangle^{\otimes n} \right|^2$$

Call this the distribution $\mathcal{D}_s$.

It is believed to be classically hard to sample from $\mathcal{D}_s$.

How should we verify that a quantum computer generated samples from $\mathcal{D}_s$?

VERIFICATION (HOG test)

1. Collect $x_1 \dots x_T$ from quantum computer.

2. Use a classical supercomputer to compute

$$\Pr_{\mathcal{D}_c}[x_1], \ \Pr_{\mathcal{D}_c}[x_2] \dots \Pr_{\mathcal{D}_c}[x_T]$$

3. Compute mean of these probabilities, call it α_c

4. Call x_i heavy if $\Pr[x_i] > \alpha_c$.

5. If $2/3$ of x_i's are heavy, accept.
   Otherwise, reject.

(Intuition: Quantum computers can reliably sample heavy elements. Classical cannot.)