MIXED STATES

A mixed state represents a probability distribution over pure states. $\{ p_i , |\psi_i\rangle \}$

p_i ↓ probability of sampling state $|\psi_i\rangle$

There's a better way to denote mixed states ....

MEASUREMENTS

Measuring a pure state $|\psi\rangle$.
(in any orthonormal basis)

Measure $|\psi\rangle = \sum_{i \in [N]} \alpha_i |i\rangle$ in basis $\{|1\rangle \dots |N\rangle\}$
"computational basis"

$\Pr["i"] = |\alpha_i|^2$

Measure $|\psi\rangle$ in $\longrightarrow$ basis $\{|v_1\rangle, |v_2\rangle \dots |v_N\rangle\}$
(any orthonormal basis)

$\Pr["v_i"] = |\langle v_i | \psi \rangle|^2$

Measure mixed state $\{ p_j, |\psi_j\rangle \}_{j\in[n]}$ in orthonormal basis $\{ |v_1\rangle, |v_2\rangle \dots |v_N\rangle \}$

$$\Pr[\text{"}i\text{"}] = \sum_j p_j \left| \langle v_i | \psi_j \rangle \right|^2$$

$$= \sum_j p_j \langle v_i | \psi_j \rangle \left(\langle v_i | \psi_j \rangle \right)^\dagger$$

$$= \sum_j p_j \langle v_i | \psi_j \rangle \langle \psi_j | v_i \rangle$$

$$= \sum_j p_j \langle v_i | \psi_j \rangle\langle \psi_j | v_i \rangle$$

$$= \langle v_i | \underbrace{\left(\sum_j p_j |\psi_j\rangle\langle\psi_j| \right)}_{\text{Density matrix } \rho} | v_i \rangle .$$

ANY measurement outcome is ONLY a function of ρ.

Density matrix of $\{ (p_j, |\psi_j\rangle) \}_{j\in[k]}$

$$= \sum_j p_j |\psi_j\rangle\langle\psi_j|$$

Measure this state in the computational basis

$$\Pr[\text{"1"}] = \begin{bmatrix} 1 & 0 & \cdots & 0 \end{bmatrix} \begin{bmatrix} \rho_{11} & \rho_{12} & & \\ \rho_{21} & & & \\ & & \ddots & \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} = \rho_{11}$$

$$\Pr[\text{"2"}] = \rho_{22}. \quad \cdots \quad \Pr[\text{"i"}] = \rho_{ii}$$

sum of prob. of seeing various values on comp. basis msmt

$$\mathrm{Tr}[\rho] = \rho_{11} + \rho_{22} + \rho_{33} \cdots\cdots \rho_{NN}$$

= sum of diagonal elements of ρ

$$= \Pr[\text{"1"}] + \Pr[\text{"2"}] + \cdots\cdots + \Pr[\text{"N"}].$$

$$= 1.$$

$$\boxed{\text{PROPERTY 1}: \ \mathrm{Tr}[\rho] = 1}$$

$$\rho = \sum_i p_i |\psi_i\rangle\langle\psi_i|$$

$$= \sum_i p_i \begin{bmatrix} \alpha_0 \\ \vdots \\ \alpha_{N-1} \end{bmatrix} \begin{bmatrix} \alpha_0^* & \cdots & \alpha_{N-1}^* \end{bmatrix}$$

$$= \sum_i p_i \begin{array}{c} \\ M_{21} \\ \\ \end{array} \begin{array}{c} \begin{array}{cc} & M_{12} \end{array} \\ \begin{bmatrix} \alpha_0\alpha_0^* & \alpha_0\alpha_1^* & \\ \alpha_1\alpha_0^* & & \\ & & \end{bmatrix} \end{array}$$

$$\rho = \rho^\dagger \quad (\text{Hermitian Matrix}).$$

$$\boxed{\text{PROPERTY 2}: \ \rho \text{ is Hermitian}}$$

Positive Semi-Definite

$\forall v \in \mathbb{C}^N, \quad \langle v|\rho|v\rangle \geq 0$

Let's assume that v is unit vector. (w.l.o.g.)

orthonormal basis : $B = \{ v, v_2, v_3 \ldots\ldots v_N \}$.

$\langle v|\rho|v\rangle = \Pr[\text{"}v\text{"}]$ when measuring in basis B.

$\boxed{\text{PROPERTY 3 : POSITIVE SEMI-DEFINITE}}$

UNITARY OPERATIONS ON MIXED STATES

$S_1 = \{(p_i, |\psi_i\rangle)\} \qquad U(S_1) \rightarrow S_2.$

$$\rho_{S_1} = \sum_i p_i |\psi_i\rangle\langle\psi_i|$$

$$\rho_{S_2} = \sum_i p_i |U(\psi_i)\rangle (|U(\psi_i)\rangle)^{\dagger}$$

$= \langle\psi_i|U^{\dagger}$

$$= \sum_i p_i U|\psi_i\rangle\langle\psi_i|U^{\dagger}$$

$$= U\left(\sum_i p_i |\psi_i\rangle\langle\psi_i|\right) U^{\dagger}$$

$$= U \rho_{S_1} U^{\dagger}$$

__Hermitian__ Matrices satisfy : for every Hermitian M

There is an orthonormal basis
$|v_1\rangle \ldots |v_N\rangle$ of $\mathbb{C}^N$ and $\boxed{\text{real}}$ ~~eigenvalues~~
(positive) "scaling factors

$\lambda_1, \ldots \lambda_N$ such that

M's action is to scale by λ_i in direction $|v_i\rangle$

i.e. $M = \sum_{j=1}^{N} \lambda_j |v_j\rangle\langle v_j|$

and "scaling" $|v_j\rangle$ by λ_j

$$\langle v_j | M | v_j \rangle = \lambda_j$$

A density matrix ρ is Hermitian,
$Tr(\rho) = 1$, $\rho \geq 0$ (p.s.d.)

$\rho \geq 0 \Rightarrow \langle v_i | \rho | v_i \rangle \geq 0 \Rightarrow \lambda_i \geq 0 \; (\forall i)$

(linear algebra fact).

For any orthonormal $|v_1\rangle \ldots |v_N\rangle$, there is
a unitary $U_{v_1 \ldots v_N}$ s.t. $\forall i$ $U|v_i\rangle \rightarrow |i\rangle$
$U^\dagger |i\rangle \rightarrow |v_i\rangle$

Given density matrix ρ with associated $\lambda_1 \dots \lambda_N$ and basis $|v_1\rangle \dots |v_N\rangle$.

Apply $U_{v_1 \dots v_N}$ to ρ, resulting in $\rho' = U \rho U^\dagger$

① [ρ' has the same λ_i's as ρ.]

② [ρ' has basis $|1\rangle \dots |N\rangle$]

ρ' action is to scale each $|i\rangle$ by λ_i

$$\langle v | \rho' | v \rangle = \langle v | U \rho U^\dagger | v \rangle$$

Simpify so that $|v\rangle = |i\rangle$.

$$\underbrace{\langle i | U}_{\langle v_i |} \; \rho \; \underbrace{U^\dagger | i \rangle}_{= |v_i\rangle} = \lambda_i$$

$$\rho' = \sum \lambda_i |i\rangle\langle i|$$

$$= \begin{bmatrix} \lambda_1 & & & 0 \\ & \lambda_2 & & \\ & & \ddots & \\ 0 & & & \lambda_N \end{bmatrix}$$

$\mathrm{Tr}[\rho'] = 1$ because ρ' is a density matrix

$\Rightarrow \lambda_1 + \lambda_2 \dots + \lambda_N = 1$.

In conclusion, Hermitian matrix ρ that is a density matrix has each $\lambda_i \geq 0$, $\sum_i \lambda_i = 1$.

So, any Density Matrix ρ with distinct* eigenvalues $\lambda_1 \dots \lambda_N$ has a canonical mixed state associated with it $\{\lambda_i, |v_i\rangle\}_{i \in [N]}$

MAXIMALLY MIXED STATE

This is a mixed state for which all of its eigenvalues / scaling factors $\lambda_1 \dots \lambda_N$ are identical

$$\sum_i \lambda_i = 1 \qquad \rho = \begin{bmatrix} \frac{1}{N} & & 0 \\ & \frac{1}{N} & \\ 0 & & \ddots & \\ & & & \frac{1}{N} \end{bmatrix} = \sum_i \underbrace{\lambda_i}_{=\frac{1}{N}} |i\rangle\langle i|$$

Game 1.

$$\left\{\tfrac{1}{2}|0\rangle, \tfrac{1}{2}|1\rangle\right\}$$

$$\rho = \frac{1}{2}\begin{bmatrix} 1 \\ 0 \end{bmatrix}\begin{bmatrix} 1 & 0 \end{bmatrix} + \frac{1}{2}\begin{bmatrix} 0 \\ 1 \end{bmatrix}\begin{bmatrix} 0 & 1 \end{bmatrix}$$

$$= \frac{1}{2}\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + \frac{1}{2}\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

$$= \frac{1}{2}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{bmatrix}$$

Maximally mixed state on 1 qubit ($N=2$)

Game 2.

$$\left\{\tfrac{1}{2}|+\rangle, \tfrac{1}{2}|-\rangle\right\}$$

$$\rho = \frac{1}{2}\begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix}\begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix} + \frac{1}{2}\begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{-1}{\sqrt{2}} \end{bmatrix}\begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} \end{bmatrix}$$

$$= \frac{1}{2}\begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix} + \frac{1}{2}\begin{bmatrix} \frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} \end{bmatrix}$$

$$= \frac{1}{2}\begin{bmatrix} \frac{1}{2}+\frac{1}{2} & \cancel{\frac{1}{2}-\frac{1}{2}} \\ \cancel{\frac{1}{2}-\frac{1}{2}} & \frac{1}{2}+\frac{1}{2} \end{bmatrix}$$

$$= \begin{bmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{bmatrix}$$

Distributions look different, but...

Identical mixed states!