Correcting both bit flip and phaseflip errors

SHOR'S 9-QUBIT CODE

$$|0\rangle \xmapsto{\text{phase flip code}} |+++\rangle = \frac{1}{2\sqrt{2}}\left(|0\rangle + |1\rangle\right)^{\otimes 3}$$

$$\Big\downarrow \text{ bit flip code}$$

$$\frac{1}{2\sqrt{2}}\left(|000\rangle + |111\rangle\right)^{\otimes 3}$$

$$|1\rangle \xmapsto{\text{phase flip code}} |---\rangle = \frac{1}{2\sqrt{2}}\left(|0\rangle - |1\rangle\right)^{\otimes 3}$$

$$\Big\downarrow \text{ bit flip code}$$

$$= \frac{1}{2\sqrt{2}}\left(|000\rangle - |111\rangle\right)^{\otimes 3}$$

This code can correct both bitflip and phaseflip errors, as long as they occur on a single qubit.

BIT-FLIP ERRORS

$$\text{Encode}(|0\rangle) = \frac{1}{2\sqrt{2}}(|000\rangle+|111\rangle)\otimes(|000\rangle+|111\rangle)\otimes(|000\rangle+|111\rangle)$$

$\downarrow$ error on 5^{th} qubit

$$\frac{1}{2\sqrt{2}}(|000\rangle+|111\rangle)\otimes(|010\rangle+|101\rangle)\otimes(|000\rangle+|111\rangle)$$

bit flip decoding $\downarrow$ bit flip decoding $\downarrow$ bit flip decoding $\downarrow$

$$\frac{1}{2\sqrt{2}}(|0\rangle+|1\rangle)\otimes(|0\rangle+|1\rangle)\otimes(|0\rangle+|1\rangle)$$

$$= \frac{1}{2\sqrt{2}}|+++\rangle$$

$\downarrow$ phaseflip decoding

$$|0\rangle$$

PHASE-FLIP ERRORS

$$\text{Encode}(|0\rangle) = \frac{1}{2\sqrt{2}}(|000\rangle+|111\rangle)\otimes(|000\rangle+|111\rangle)\otimes(|000\rangle+|111\rangle)$$

$\downarrow$ error on 5th qubit

$$\frac{1}{2\sqrt{2}}(|000\rangle+|111\rangle)\otimes(|000\rangle-|111\rangle)\otimes(|000\rangle+|111\rangle)$$

bit flip decoding $\downarrow$ bit flip decoding $\downarrow$ bit flip decoding $\downarrow$

$$\frac{1}{2\sqrt{2}}(|0\rangle+|1\rangle)\otimes(|0\rangle-|1\rangle)\otimes(|0\rangle+|1\rangle)$$

$$= \frac{1}{2\sqrt{2}}|+-+\rangle$$

$\downarrow$ phaseflip decoding

$|0\rangle$

Claim: SHOR'S 9-qubit code can handle more than just bit-flip and phase-flip errors.

Can handle any UNITARY ERRORS.

Every 2×2 matrix can be written as a linear combination of Pauli matrices.

why? Linear independence

$$\forall M \in \mathbb{C}^{2\times 2}, \quad M = \alpha \mathbb{I} + \beta X + \gamma Y + \delta Z$$

where $\alpha, \beta, \gamma, \delta \in \mathbb{C}$

(recall $Y = iXZ$)

Suppose unitary error U occurs on $|\phi\rangle$

$\rightarrow$ resulting erroneous state.

$$|\tilde{\phi}\rangle = U|\phi\rangle$$

$$= (\alpha \mathbb{I} + \beta X + \gamma Y + \delta Z)|\phi\rangle$$

$$= \alpha|\phi\rangle + \beta X|\phi\rangle + \gamma Y|\phi\rangle + \delta Z|\phi\rangle$$

Now apply correction to $|\tilde{\phi}\rangle$

Recall that C corrects from

- no error, so : $C|\phi\rangle|0^n\rangle = |\phi\rangle|aux_0\rangle$
- X errors, so $CX|\phi\rangle = |\phi\rangle|aux_1\rangle$
- Z errors, so $CZ|\phi\rangle = |\phi\rangle|aux_2\rangle$
- XZ errors, so $CXZ|\phi\rangle = |\phi\rangle|aux_3\rangle$

$$C|\tilde{\phi}\rangle|0^n\rangle$$

$$= C\left[U|\phi\rangle|0^n\rangle\right]$$

$$= C\alpha|\phi\rangle + C\beta X|\phi\rangle + C\gamma Y|\phi\rangle + C\delta Z|\phi\rangle$$

$\hookrightarrow Y = iXZ$

$$= \alpha|\phi\rangle|aux_1\rangle + \beta|\phi\rangle|aux_2\rangle + i\gamma|\phi\rangle|aux_3\rangle + \delta|\phi\rangle|aux_4\rangle$$

$$= |\phi\rangle \otimes \left(\alpha|aux_1\rangle + \beta|aux_2\rangle + i\gamma|aux_3\rangle + \delta|aux_4\rangle\right)$$

$$= |\phi\rangle \otimes |aux\rangle$$

What about non-unitary errors?

for example, qubits get dropped / lost / measured.

→ Every quantum operation is equivalent to a UNITARY on a larger Hilbert space

Can write ANY operation as

$$|\phi\rangle \mapsto E|\phi\rangle|0^n\rangle \quad \text{where } E \text{ is unitary}$$

Can decompose E as

$$E = \alpha\, \mathbb{I} \otimes E_1 + \beta X \otimes E_2 + \gamma Y \otimes E_3 + \delta Z \otimes E$$

$$E|\phi\rangle|0^n\rangle = \alpha|\phi\rangle \otimes E_1|0^n\rangle + \beta X|\phi\rangle \otimes E_2|0^n\rangle$$
$$+ \gamma Y|\phi\rangle \otimes E_3|0^n\rangle + \delta Z|\phi\rangle \otimes E_4|0^n\rangle$$

Applying the correction circuit, 4

$$CE|\phi\rangle|0^n\rangle = \alpha\, C|\phi\rangle \otimes E_1|0^n\rangle + \beta\, CX|\phi\rangle \otimes E_2|0^n\rangle$$
$$+ \gamma\, CY|\phi\rangle \otimes E_3|0^n\rangle + \delta\, CZ|\phi\rangle \otimes E_4|0^n\rangle$$

$$= \alpha|\phi\rangle \otimes E_1|0^n\rangle + \beta|\phi\rangle \otimes E_2|0^n\rangle$$
$$+ i\gamma|\phi\rangle \otimes E_3|0^n\rangle + \delta|\phi\rangle \otimes E_4|0^n\rangle$$

$$= |\phi\rangle \otimes \left[\alpha E_1|0^n\rangle + \beta E_2|0^n\rangle + i\gamma E_3|0^n\rangle + \delta E_4|0^n\rangle\right]$$

$$= |\phi\rangle \otimes |aux\rangle$$

recovers $|\phi\rangle$, as desired