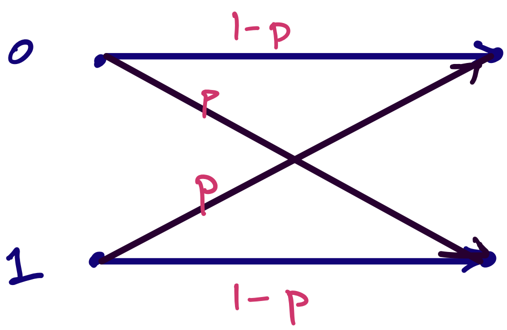
CLASSICAL ERROR CORRECTION

Modeling Classical Errors

Simplest case: transmitting a single bit

if either passes through channel correctly, or gets flipped

Binary symmetric channel



Each bit is flipped independently with probability p.

$$m \xrightarrow{\text{Encode}(m)} c \Longrightarrow \tilde{c} \xrightarrow{\text{Decode}(\tilde{c})} \tilde{m}$$

m	c		$\tilde{c}$
message	codeword	channel introduces errors	erroneous codeword

Goal: Want $\tilde{m} = m$

Type of code used, and whether $\tilde{m} = m$ is even achievable, depends on the type of errors that the channel introduces.

Classical error correcting code:

Repetition code: $0 \xrightarrow{\text{Encode}} 000\ldots0$ (n times)

$1 \xrightarrow{\text{Encode}} 111\ldots1$ (n times)

Decode $(\tilde{c})$: output majority (bits in $\tilde{c}$)

As long as $p < \frac{1}{2}$, w.h.p., decoded bit $\tilde{m}$ = encoded bit m

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Modeling Quantum Errors

Qubit $|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$

→ "Bit flip" is not the only error

Channel can change $|\psi\rangle$ to $|\phi\rangle$ arbitrarily (rotate by arbitrary angle θ, infinitely many possible θ).

So infinitely many errors?

→ cannot encode $|\psi\rangle$ as $|\psi\rangle \otimes |\psi\rangle \otimes |\psi\rangle$ (violates no-cloning)

→ cannot measure erroneous codeword in standard basis (this measurement will destroy phase information in the originally encoded qubit)

NOT ALL HOPE IS LOST.

Turns out, we need to care about only three types of errors.

1) BIT-FLIP ERROR

$$X|0\rangle \rightarrow |1\rangle \qquad X|1\rangle \rightarrow |0\rangle$$

$$X(\alpha|0\rangle + \beta|1\rangle) \rightarrow \beta|0\rangle + \alpha|1\rangle$$

2) PHASE-FLIP ERROR

$$Z|0\rangle \rightarrow |0\rangle \qquad Z|1\rangle \rightarrow -|1\rangle$$

$$Z(\alpha|0\rangle + \beta|1\rangle) \rightarrow \alpha|0\rangle - \beta|1\rangle$$

3) BITFIP AND PHASE-FLIP ERROR

$$XZ|0\rangle \rightarrow |1\rangle \qquad XZ|1\rangle \rightarrow -|0\rangle$$

$$XZ(\alpha|0\rangle + \beta|1\rangle) \rightarrow -\beta|0\rangle + \alpha|1\rangle$$

$$= \beta|0\rangle - \alpha|1\rangle$$

How to correct bit-flip errors?

Encode: $|0\rangle \rightarrow |000\rangle$

$|1\rangle \rightarrow |111\rangle$

$$\alpha|0\rangle + \beta|1\rangle \rightarrow \alpha|000\rangle + \beta|111\rangle$$

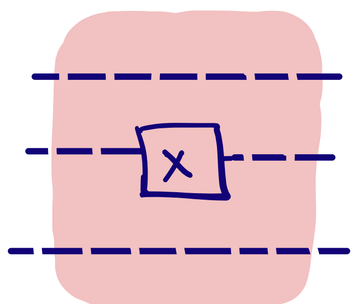
In-class: 1) Does this violate no-cloning?

2) How do you implement the unitary

$$U : (\alpha|0\rangle + \beta|1\rangle) \otimes |00\rangle \rightarrow \alpha|000\rangle + \beta|111\rangle$$

channel

$|\psi\rangle = \alpha|0\rangle + \beta|1\rangle \xrightarrow{\text{Encode}} |\phi\rangle = \alpha|000\rangle + \beta|111\rangle$



$|\tilde{\phi}\rangle = \alpha|010\rangle + \beta|101\rangle \xrightarrow{\text{Decode}} |\tilde{\psi}\rangle$

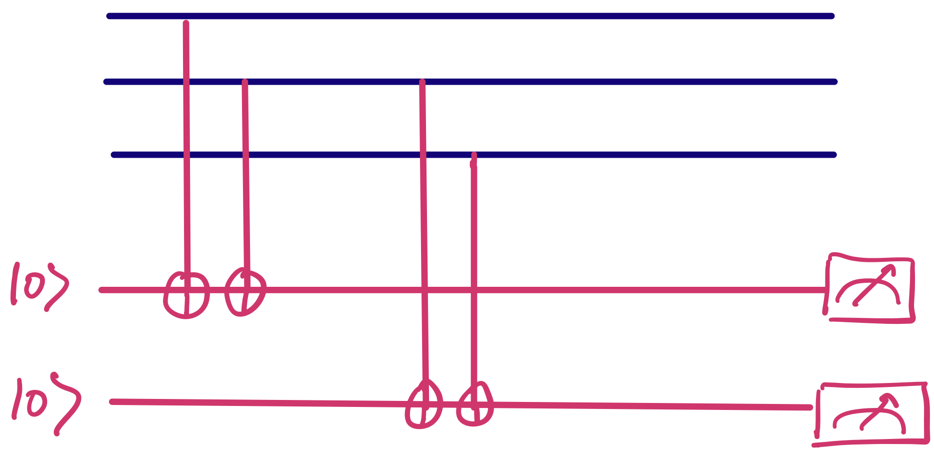
Goal: Want $|\tilde{\psi}\rangle = |\psi\rangle$

Can you measure $|\tilde{\phi}\rangle$ in computational basis?

No! Collapses to 010 or 101, destroying α/β info.

Compute SYNDROMES

$|\tilde{\Phi}\rangle = \alpha|010\rangle + \beta|101\rangle$



$$\alpha|010\rangle|11\rangle + \beta|101\rangle|11\rangle = (\alpha|010\rangle + \beta|101\rangle) \otimes |11\rangle$$

measure

obtain syndrome bits $(1,1)$

This means first qubit $\neq$ second qubit

and second qubit $\neq$ third qubit

$\Rightarrow$ error most likely on second qubit

So, flip second qubit (i.e., apply a Z gate)

syndrome bits

a	b	error
0	0	no error
0	1	3rd qubit
1	0	1st qubit
1	1	2nd qubit

Don't need to measure these.

Can correct errors by flipping conditioned on these qubits.

How to correct phase-flip errors?

Recall the error:
$$Z|0\rangle \rightarrow |0\rangle$$
$$Z|1\rangle \rightarrow -|1\rangle$$

$$Z|+\rangle = Z\left(\frac{|0\rangle+|1\rangle}{\sqrt{2}}\right) = \frac{|0\rangle-|1\rangle}{\sqrt{2}} = |-\rangle$$

$$Z|-\rangle = Z\left(\frac{|0\rangle-|1\rangle}{\sqrt{2}}\right) = \frac{|0\rangle+|1\rangle}{\sqrt{2}} = |+\rangle$$

"Flips" $|+\rangle$ to $|-\rangle$ and $|-\rangle$ to $|+\rangle$

So is a bit-flip error in the Hadamard basis!

Q: How to correct bit-flip errors in Hadamard basis?

A: Repetition code in the Hadamard basis!

Encode: $|0\rangle \rightarrow |{+}{+}{+}\rangle$

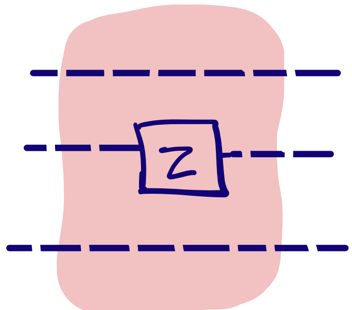
$|1\rangle \rightarrow |{-}{-}{-}\rangle$

$$\alpha|0\rangle + \beta|1\rangle \longrightarrow \alpha|{+}{+}{+}\rangle + \beta|{-}{-}{-}$$

In-class: How do you implement this unitary?

$|\psi\rangle = \alpha|0\rangle + \beta|1\rangle \xmapsto{\text{Encode}} |\phi\rangle = \alpha|{+}{+}{+}\rangle + \beta|{-}{-}{-}\rangle$

Channel



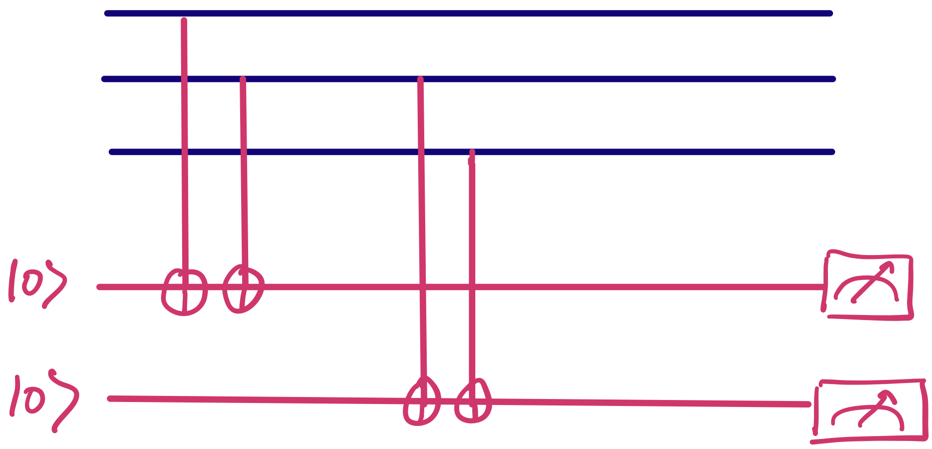
$|\tilde{\phi}\rangle = \alpha|{+}{-}{+}\rangle + \beta|{-}{+}{-}\rangle \xmapsto{\text{Decode}} |\tilde{\psi}\rangle$

Goal: Want $|\tilde{\psi}\rangle = |\psi\rangle$.

First apply $H^{\otimes 3}(|\tilde{\phi}\rangle) \longmapsto \alpha|010\rangle + \beta|101\rangle$

Then, as before, compute SYNDROMES and decode.

$|\tilde{\phi}\rangle = \alpha|010\rangle + \beta|101\rangle$



Correcting both bit flip and phaseflip errors

SHOR'S 9-QUBIT CODE

$$|0\rangle \xmapsto{\text{phase flip code}} |{+}{+}{+}\rangle = \frac{1}{2\sqrt{2}}(|0\rangle + |1\rangle)^{\otimes 3}$$

$$\Big\downarrow \text{ bit flip code}$$

$$\frac{1}{2\sqrt{2}}(|000\rangle + |111\rangle)^{\otimes 3}$$

$$|1\rangle \xmapsto{\text{phase flip code}} |{-}{-}{-}\rangle = \frac{1}{2\sqrt{2}}(|0\rangle - |1\rangle)^{\otimes 3}$$

$$\Big\downarrow \text{ bit flip code}$$

$$= \frac{1}{2\sqrt{2}}(|000\rangle - |111\rangle)^{\otimes 3}$$

This code can correct both bitflip and phaseflip errors, as long as they occur on a single qubit.

BIT-FLIP ERRORS

$$\text{Encode}(|0\rangle) = \frac{1}{2\sqrt{2}}\left(|000\rangle+|111\rangle\right)\otimes\left(|000\rangle+|111\rangle\right)\otimes\left(|000\rangle+|111\rangle\right)$$

$\downarrow$ error on 5^{th} qubit

$$\frac{1}{2\sqrt{2}}\left(|000\rangle+|111\rangle\right)\otimes\left(|010\rangle+|101\rangle\right)\otimes\left(|000\rangle+|111\rangle\right)$$

bit flip decoding $\downarrow$ bit flip decoding $\downarrow$ bit flip decoding $\downarrow$

$$\frac{1}{2\sqrt{2}}\left(|0\rangle+|1\rangle\right)\otimes\left(|0\rangle+|1\rangle\right)\otimes\left(|0\rangle+|1\rangle\right)$$

$$= \frac{1}{2\sqrt{2}}|{+}{+}{+}\rangle$$

$\downarrow$ phaseflip decoding

$$|0\rangle$$

PHASE-FLIP ERRORS

$$\text{Encode}(|0\rangle) = \frac{1}{2\sqrt{2}}(|000\rangle+|111\rangle)\otimes(|000\rangle+|111\rangle)\otimes(|000\rangle+|111\rangle)$$

$\downarrow$ error on 5^{th} qubit

$$\frac{1}{2\sqrt{2}}(|000\rangle+|111\rangle)\otimes(|000\rangle-|111\rangle)\otimes(|000\rangle+|111\rangle)$$

bit flip decoding $\downarrow$ bit flip decoding $\downarrow$ bit flip decoding $\downarrow$

$$\frac{1}{2\sqrt{2}}(|0\rangle+|1\rangle)\otimes(|0\rangle-|1\rangle)\otimes(|0\rangle+|1\rangle)$$

$$= \frac{1}{2\sqrt{2}}|+-+\rangle$$

$\downarrow$ phaseflip decoding

$$|0\rangle$$