LECTURE 7 September 12^{th}, 2023

PART I Fundamental Concepts & Applications in Quantum Information

TODAY Bell's Theorem & the CHSH game

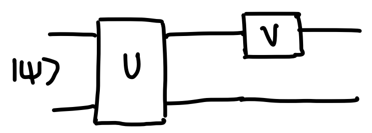
RECAP Fundamental Concepts in Quantum Information

State of multi-qubit systems $\alpha_{00}|00\rangle + \alpha_{01}|01\rangle + \alpha_{10}|10\rangle + \alpha_{11}|11\rangle$

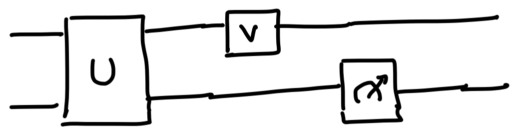
Entanglement Not of the form $|\psi\rangle \otimes |\phi\rangle$

e.g the Bell State / EPR pair $\dfrac{|00\rangle + |11\rangle}{\sqrt{2}}$

Unitary Transformations & Quantum Circuits



Partial Measurements & Mixed State



Mixed State

Partial Measurements for Qudits

Suppose Alice and Bob have entangled qutrits

$$\alpha_{11}|11\rangle + \alpha_{12}|12\rangle + \alpha_{11}|13\rangle + \alpha_{21}|21\rangle + \dots + \alpha_{33}|33\rangle$$

$$\mathbb{P}[\text{ Alice's measurement is "1" }] = |\alpha_{11}|^2 + |\alpha_{12}|^2 + |\alpha_{13}|^2 := p_1$$

& state becomes $\dfrac{\alpha_{11}|11\rangle + \alpha_{12}|12\rangle + \alpha_{13}|13\rangle}{\sqrt{p_1}}$

and so on ....

Fact to Remember : Measuring qubits one by one gives the same outcome as measuring both qubits ← Can do it in any order

Exercise (in-class) Give a circuit to prepare the Bell state starting from the state $|++\rangle$

EPR Paradox Alice & Bob prepare the Bell state $\frac{|00\rangle + |11\rangle}{\sqrt{2}}$

They each take one of the qubits & go far away

① If Alice measures her qubit in $\{|0\rangle, |1\rangle\}$ basis
Bob's qubit changes to whatever was measured

Is this faster-than-light communication?

No! Alice & Bob learn a random bit

② But Alice learns Bob's outcome instanteously

Alice & Bob have two copies of a classical coin tossed by Charlie Local Hidden Variable
They only look at it when they are far away
No violation of classical physics!

Local Hidden Variable Theory

③ Alice can try to send a bit by measuring in
either $\{|0\rangle, |1\rangle\}$ basis
or $\{|+\rangle, |-\rangle\}$ basis

EPR pair $\frac{|00\rangle + |11\rangle}{\sqrt{2}} = \frac{|++\rangle + |--\rangle}{\sqrt{2}}$

Bob's local state is $\rho_0 = \begin{cases} 50\% \text{ chance } |0\rangle \\ 50\% \text{ chance } |1\rangle \end{cases}$ or $\rho_1 = \begin{cases} 50\% \text{ chance } |+\rangle \\ 50\% \text{ chance } |-\rangle \end{cases}$

Nothing Bob does can distinguish ρ_0 from ρ_1

The question of whether quantum mechanics is a local hidden variable theory or not went unsolved for 30 years

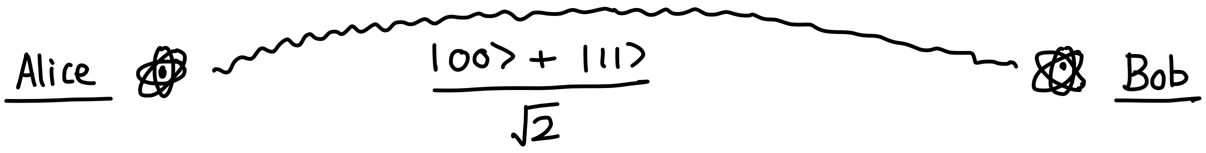
Bell's Theorem No local hidden variable theory can be compatible with quantum mechanics

Bell in 1964 designed an experiment called the Bell test such that the predictions of quantum mechanics differ from the predictions of any local hidden variable theory

CHSH game was a simplification of Bell's experiment devised in 1970s by Clauser, Horne, Shimony and Holt ← One of the major discoveries in Quantum Mechanics!!

CHSH game Alice & Bob prepare the Bell state $\frac{|00\rangle + |11\rangle}{\sqrt{2}}$

They each take one of the qubits & go far away say Alice goes to Mars & Bob goes to Jupiter

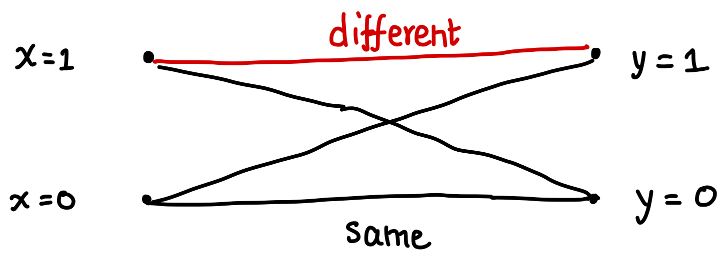


They are both issued a challenge by a referee as follows:

- Challenge to Alice is $x \in \{0,1\}$ and to Bob is $y \in \{0,1\}$ where x and y are independent random bits
- Referee puts the challenge in a box & Alice & Bob look at it at the same time
- They are both given 10 seconds to respond with a bit and Mars is at least 30 light minutes from Jupiter so no time for Alice to secretly communicate with Bob
- The boxes collect their responses and fly back to the referee
- They win the game if Alice's response bit $a \in \{0,1\}$ and Bob's response $b \in \{0,1\}$ satisfy the following

$$a \oplus b = x \wedge y$$

Another way of visualizing what happens in the game is via the following graph



Referee chooses a random edge and Alice & Bob's bit a & b should be different for the red edge and same otherwise in order to win the game

What is the maximum winning probability for Alice & Bob?

Deterministic Strategies Suppose Alice and Bob use deterministic strategies

Alice's answer a is a fixed function $a(x)$ of her question. Similarly, Bob's answer is also a function $b(y)$

For instance, Say Alice and Bob always answer 0

So, $a(x) = 0 \quad \forall\, x \in \{0,1\}$

$b(y) = 0 \quad \forall\, y \in \{0,1\}$

Then, they win if they get any of the black edges

$$\Rightarrow \quad \mathbb{P}[\text{winning}] = \frac{3}{4}$$

One can try all possible functions $a(x)$ and $b(y)$ and see that the maximum winning probability is $\frac{3}{4}$

Local Hidden Variable Strategy Suppose Alice & Bob are described by local hidden variables

This means that $\exists$ an underlying random variable λ such that

① Before the game, λ is sampled from some probability distribution $\mathcal{L}$

② Questions (x,y) sampled independently of λ

③ Alice's answer is a function $a(x,\lambda)$

④ Bob's answer is a function $b(y,\lambda)$

Note: One can think of λ as shared random coins

What is the maximum winning probability for Alice & Bob?

The ability to use a hidden random variable λ does not help Alice & Bob: their maximum winning probability is $3/4$

Proof $\quad \mathbb{P}[\text{win}] = \sum_{\lambda} \mathbb{P}[\lambda] \cdot \mathbb{P}[\text{win} \mid \lambda]$

But if λ is fixed, Alice and Bob's answers are deterministic functions of their questions only meaning $\mathbb{P}[\text{win} \mid \lambda] \leq \frac{3}{4}$

Therefore, $\mathbb{P}[\text{win}] \leq \sum_{\lambda} \mathbb{P}[\lambda] \cdot \frac{3}{4} \leq \frac{3}{4}$

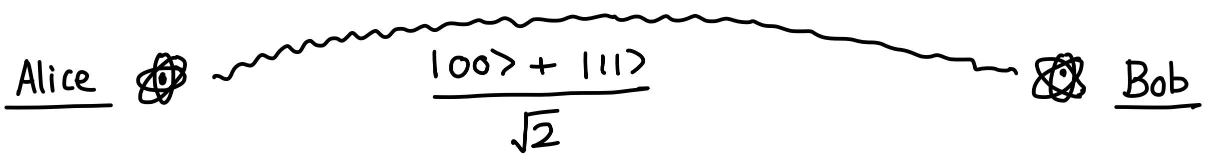
Einstein would have predicted that Alice & Bob cannot win with probability greater than $\frac{3}{4}$ in the CHSH game!!

What does Quantum Mechanics predict?

There exists a quantum strategy involving quantum entanglement where Alice & Bob win with probability ≈ 85%

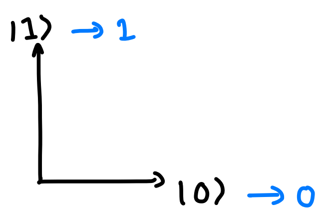
This gives an experiment to rule out local hidden variable theories

Quantum strategy

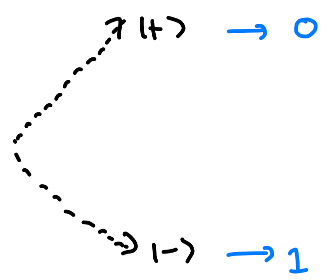


- Alice and Bob hold one qubit that jointly form an EPR pair
- Alice chooses either basis A_0 if $x=0$ to measure her qubit
  or basis A_1 if $x=1$ and interprets it as 0 or 1
- Bob chooses either basis B_0 if $y=0$ to measure his qubit
  or basis B_1 if $y=1$ and interprets it as 0 or 1

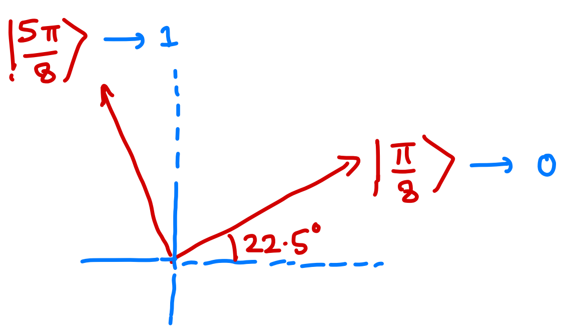
A_0 if $x=0$



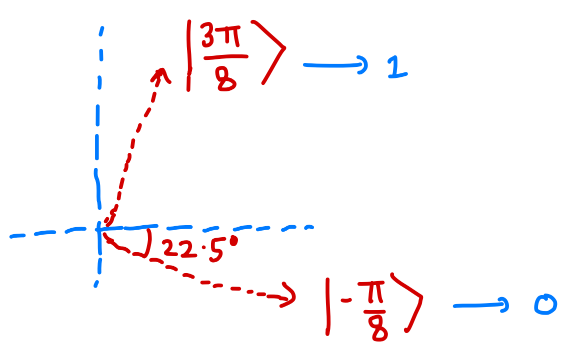
A_1 if $x=1$



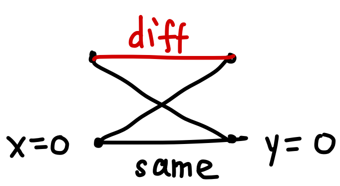
B_0 if $y=0$



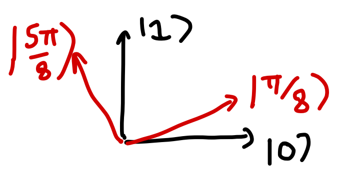
B_1 if $y=1$



How well does this strategy do?



① Suppose $x=0$ and $y=0$
Alice measures in $\{|0\rangle, |1\rangle\}$ basis
& Bob measures in $\{|\frac{\pi}{8}\rangle, |\frac{5\pi}{8}\rangle\}$ basis



In order to win, Alice & Bob's answer must match

Since the order of measurement does not matter,
when Alice measures $|0\rangle$ with probability $1/2$ & joint state is $|0\rangle \otimes |0\rangle$

To win, Bob must measure his qubit & get the $|\frac{\pi}{8}\rangle$ outcome

Since his qubit is now in $|0\rangle$ state, he gets this outcome with probability

$$|\langle 0|\tfrac{\pi}{8}\rangle|^2 = \cos^2(\pi/8) = 0.853$$

when Alice measures $|1\rangle$ with probability $1/2$ & joint state is $|1\rangle \otimes |1\rangle$

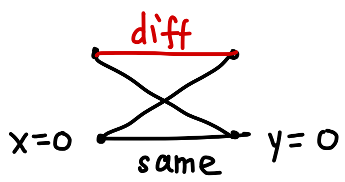
To win, Bob must measure his qubit & get the $|\frac{5\pi}{8}\rangle$ outcome

Since his qubit is now in $|1\rangle$ state, he gets this outcome with probability

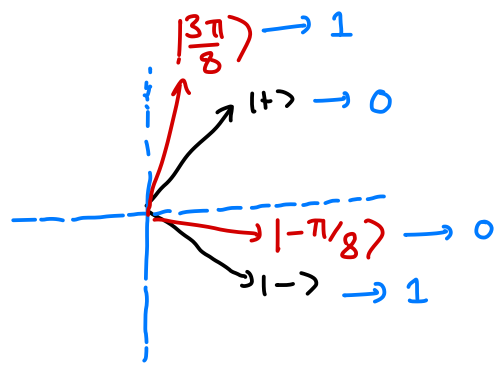
$$|\langle 1|\tfrac{5\pi}{8}\rangle|^2 = \cos^2(\pi/8) = 0.8535$$

In either case, they win with probability $\cos^2(\frac{\pi}{8}) \approx 0.8535$

② Let's take a different case



① Suppose $x=1$ and $y=1$
Alice measures in $\{|\pm\rangle\}$ basis
& Bob measures in $\{|-\frac{\pi}{8}\rangle, |\frac{3\pi}{8}\rangle\}$ basis



In order to win, Alice & Bob's answer must differ

when Alice measures $|+\rangle$ with probability $1/2$ & joint state is $|+\rangle \otimes |+\rangle$

To win, Bob must measure his qubit & get the $|\frac{3\pi}{8}\rangle$ outcome

Since his qubit is now in $|+\rangle$ state, he gets this outcome with probability

$$|\langle +|\tfrac{3\pi}{8}\rangle|^2 = \cos^2(\pi/8) = 0.853$$

when Alice measures $|-\rangle$ with probability $\frac{1}{2}$ & joint state is $|-\rangle \otimes |-\rangle$

To win, Bob must measure his qubit & get the $\left|\frac{-\pi}{8}\right\rangle$ outcome

Since his qubit is now in $|-\rangle$ state, he gets this outcome with probability

$$|\langle -| -\tfrac{\pi}{8}\rangle|^2 = \cos^2(\pi/8) = 0.853$$

Checking the other two cases, you can see that they always win with probability $\cos^2\left(\frac{\pi}{8}\right) \approx 0.8535$

This shows that there is quantum advantage in the CHSH game

It turns out that $\cos^2\left(\frac{\pi}{8}\right)$ is the best win probability for quantum strategies

This is called Tsirelson's theorem and we won't prove it in this course

Quantum advantage in the CHSH game comes from shared entanglement

Local measurements on entangled states give rise to correlations that are stronger than any classical correlations

These correlations are often called non-local

Experimental Confirmation of Bell's Theorem

Since 1970s many experiments conducted and all demonstrate winning probabilities of more than 80% which cannot be explained with Local Hidden Variable theories.

Conclusion: ① Quantum Mechanics is fundamentally a non-classical theory & Nature seems to be quantum mechanical

② Nature is inherently probabilistic

In 2015, a "loophole free" Bell test was conducted for the first time
This avoids (1) Locality loophole
(2) Detection loophole

2022 Nobel Prize in Physics

Applications: Randomness generation, Verifying quantum computers, Quantum cryptography

NEXT TIME Quantum Teleportation