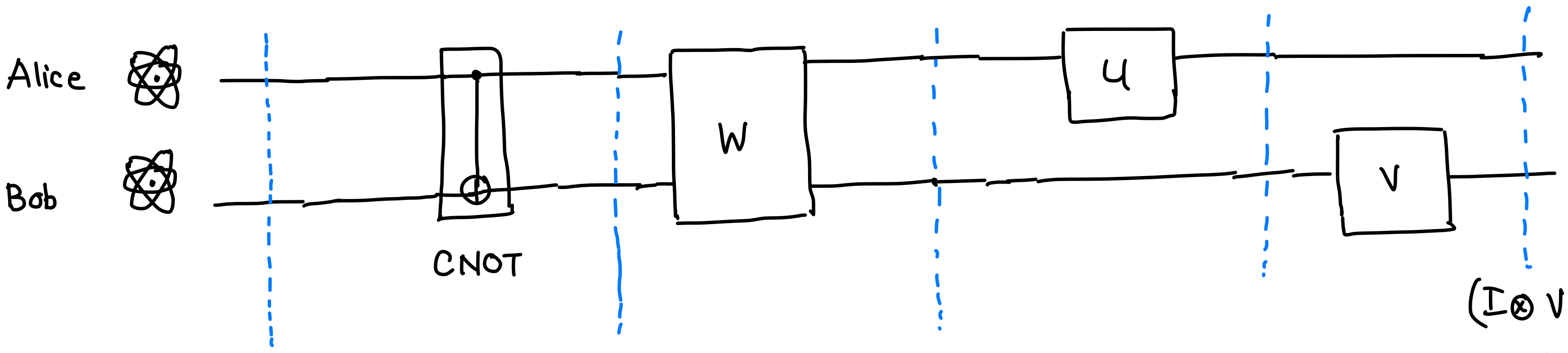
LECTURE 6 September 7^{th}, 2023

PART I Fundamental Concepts in Quantum Information — Multi-qubit systems
- superposition & entanglement
- quantum circuits

TODAY Partial Measurements & "Spooky Action at a distance"

RECAP

$(U \otimes I_{2\times 2})|\xi\rangle$



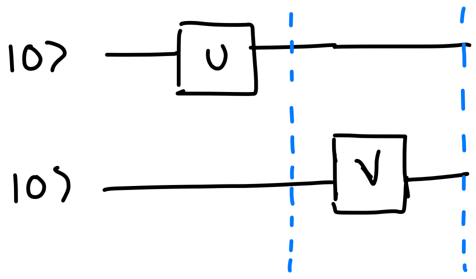
$|\psi\rangle \otimes |\phi\rangle$ $\qquad$ CNOT $|\psi\rangle\otimes|\phi\rangle$ $\qquad$ $|\xi\rangle = W\,(\text{CNOT}\,|\psi\rangle\otimes|\phi\rangle)$

$(I\otimes V)(U\otimes I)|\xi\rangle$
$= U \otimes V\,|\xi\rangle$

Order does not matter

Why? Suppose Alice & Bob had unentangled particles

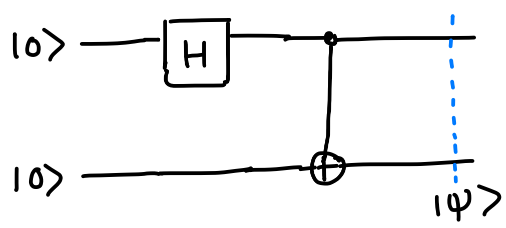
e.g.



$U|0\rangle\otimes|0\rangle$ $\qquad$ $U|0\rangle \otimes V|0\rangle = (U\otimes V)|00\rangle$

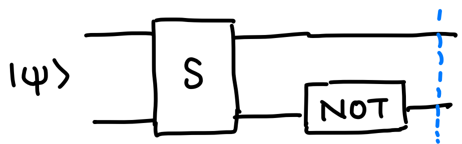
Same happens on other basis vectors $|01\rangle, |10\rangle, |11\rangle$

Exercises (in-class) (1)



What is the state at the end?

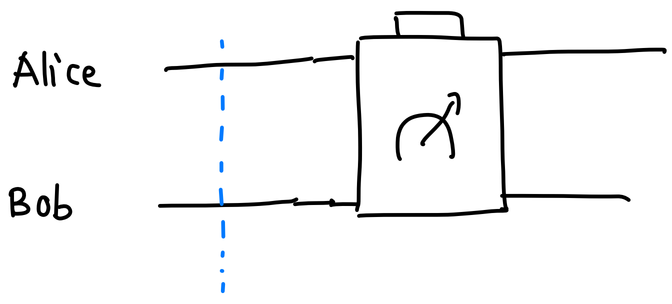
(2) Let $S = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$ What is the state $(I\otimes \text{NOT})\cdot S \cdot |\psi\rangle$



What if we measure one of the qubits?

Born's Rule for Partial Measurements

Suppose Alice & Bob have 2 photons in an entangled state possibly



$\alpha_{00}|00\rangle + \alpha_{01}|01\rangle$
$+ \alpha_{10}|10\rangle + \alpha_{11}|11\rangle$

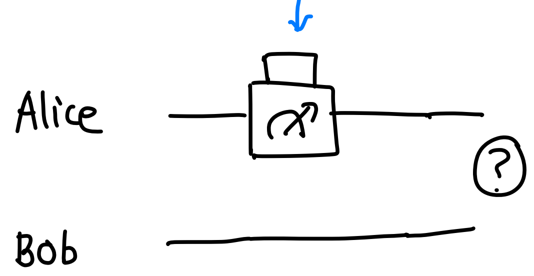
Measure both particles in $\{|0\rangle, |1\rangle\}$ basis

$\mathbb{P}[\text{outcome "00"}] = |\alpha_{00}|^2$ & state becomes $|00\rangle$

..... and so on ...

Suppose only Alice's photon is measured in $\{|0\rangle, |1\rangle\}$ basis

What are the outcome probabilities?
What is the collapsed state?



<u>Answer:</u> $\mathbb{P}[\text{ outcome is "}|0\rangle\text{"}] = |\alpha_{00}|^2 + |\alpha_{01}|^2 := p_0$

sum of squared amplitudes of terms where Alice's qubit is $|0\rangle$

State after outcome "$|0\rangle$"

$$\frac{\alpha_{00}|00\rangle + \alpha_{01}|01\rangle}{\sqrt{|\alpha_{00}|^2 + |\alpha_{01}|^2}} = \begin{bmatrix} \frac{\alpha_{00}}{\sqrt{|\alpha_{00}|^2 + |\alpha_{01}|^2}} \\ \frac{\alpha_{01}}{\sqrt{|\alpha_{00}|^2 + |\alpha_{01}|^2}} \\ 0 \\ 0 \end{bmatrix}$$

Similarly, $\mathbb{P}[\text{ outcome is "}|1\rangle\text{"}] = 1 - p_0 = |\alpha_{10}|^2 + |\alpha_{11}|^2 := p_1$

and state collapses to $\dfrac{\alpha_{10}}{\sqrt{|\alpha_{10}|^2 + |\alpha_{11}|^2}} + \dfrac{\alpha_{11}}{\sqrt{|\alpha_{10}|^2 + |\alpha_{11}|^2}}$

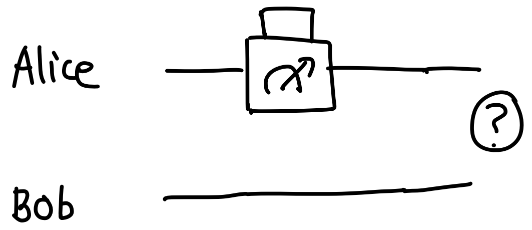
<u>Observe:</u> The collapsed state is unentangled

Outcome "$|0\rangle$": $|0\rangle \otimes \left(\dfrac{\alpha_{00}}{\sqrt{|\alpha_{00}|^2 + |\alpha_{01}|^2}} |0\rangle + \dfrac{\alpha_{01}}{\sqrt{|\alpha_{00}|^2 + |\alpha_{01}|^2}} |1\rangle \right)$

Outcome "$|1\rangle$" : $\quad |1\rangle \otimes \left(\dfrac{\alpha_{10}}{\sqrt{|\alpha_{10}|^2+|\alpha_{11}|^2}} |0\rangle + \dfrac{\alpha_{11}}{\sqrt{|\alpha_{10}|^2+|\alpha_{11}|^2}} |1\rangle \right)$

<u>Subtlety</u> We know what the state is if we see the measurement outcome?

But how do we describe the post-measurement state if we haven't observed the outcome?



← This is what is called a <u>mixed state</u>

which is a probability distribution over quantum states

What we have been looking at so far are pure quantum states

In some sense, a mixed state is the true quantum state of a system

We will mainly study pure quantum states since in quantum computing one can assume WLOG that measurement only happens at the end

We might talk about how to represent mixed states later in the course

(Mixed) State ? is :
$$\begin{cases} p_0 \text{ chance} : \dfrac{\alpha_{00}}{\sqrt{p_0}} |0\rangle + \dfrac{\alpha_{01}}{\sqrt{p_0}} |1\rangle & \text{(Case "0")} \\[2ex] p_1 \text{ chance} : \dfrac{\alpha_{10}}{\sqrt{p_1}} |0\rangle + \dfrac{\alpha_{11}}{\sqrt{p_1}} |1\rangle & \text{(Case "1")} \end{cases}$$

What happens if we measure the 2^{nd} qubit ?

<u>In case "0"</u> : $\mathbb{P}$[Bob's measurement outcome is "0"] $= \left| \dfrac{\alpha_{00}}{\sqrt{p_0}} \right|^2 = \dfrac{|\alpha_{00}|^2}{p_0}$

and state collapses to $\dfrac{\frac{\alpha_{00}}{\sqrt{p_0}} |00\rangle}{\sqrt{\left|\frac{\alpha_{00}}{\sqrt{p_0}}\right|^2}} = \boxed{\dfrac{\alpha_{00}}{|\alpha_{00}|}} |00\rangle$

Global phase

$$\mathbb{P}[\text{ Bob's measurement outcome is "1"}] = \left|\frac{\alpha_{01}}{\sqrt{p_0}}\right|^2 = \frac{|\alpha_{01}|^2}{p_0}$$

and state collapses to $\frac{\alpha_{01}}{|\alpha_{01}|}|01\rangle$

Similar for case "1"

$$\mathbb{P}[\text{ outcomes are "00"}] = p_0 \cdot \frac{|\alpha_{00}|^2}{p_0} = |\alpha_{00}|^2$$

& state collapses to $\frac{\alpha_{00}}{|\alpha_{00}|}|00\rangle$

$$\mathbb{P}[\text{ outcomes are "01"}] = p_0 \cdot \frac{|\alpha_{01}|^2}{p_0} = |\alpha_{01}|^2$$

& state collapses to $\frac{\alpha_{01}}{|\alpha_{01}|}|01\rangle$

Another subtle point Suppose Alice and Bob have an unentangled two qubit state

$$|\psi\rangle \otimes |\phi\rangle$$

Suppose Alice walks away, what's the state of Bob's qubit ?

Answer : $|\phi\rangle$

What if Alice & Bob had a two qubit entangled state

$$|\psi\rangle \in \mathbb{C}^4$$

If Alice walks away, what's the state of Bob's qubit ?

We can describe in terms of a mixed state

Alice measures her qubit to be "0": p_0 chance & state collapses to

$$|0\rangle \otimes \left(\frac{\alpha_{00}}{\sqrt{p_0}} |0\rangle + \frac{\alpha_{01}}{\sqrt{p_0}} |1\rangle \right)$$

Alice measures her qubit to be "1": p_1 chance & state collapses to

$$|1\rangle \otimes \left(\frac{\alpha_{10}}{\sqrt{p_1}} |0\rangle + \frac{\alpha_{11}}{\sqrt{p_1}} |1\rangle \right)$$

State of Bob's qubit can only be described by the mixed state:

$$p_0 \text{ chance} : \frac{\alpha_{00}}{\sqrt{p_0}} |0\rangle + \frac{\alpha_{01}}{\sqrt{p_1}} |1\rangle$$

$$p_1 \text{ chance} : \frac{\alpha_{10}}{\sqrt{p_1}} |0\rangle + \frac{\alpha_{11}}{\sqrt{p_1}} |1\rangle$$

Any measurement that Bob performs on this mixed state will give the same outcome
This is because the order of measurements does not matter

Partial Measurements for Qudits

Suppose Alice and Bob have entangled qutrits

$$\alpha_{11}|11\rangle + \alpha_{12}|12\rangle + \alpha_{11}|13\rangle + \alpha_{21}|21\rangle + \dots + \alpha_{33}|33\rangle$$

$$\mathbb{P}[\text{Alice's measurement is "1"}] = |\alpha_{11}|^2 + |\alpha_{12}|^2 + |\alpha_{13}|^2 := p_1$$

& state becomes $\dfrac{\alpha_{11}|11\rangle + \alpha_{12}|12\rangle + \alpha_{13}|13\rangle}{\sqrt{p_1}}$

and so on ....

EPR Paradox

Suppose Alice & Bob have an EPR pair

$$\left.\begin{array}{l} \text{Alice} \quad \cdot \\ \\ \text{Bob} \quad \cdot \end{array}\right\} \begin{array}{l} \text{Bell State} \\ \frac{1}{\sqrt{2}}|00\rangle + \frac{1}{\sqrt{2}}|11\rangle \end{array}$$

Alice can walk far away and the particles are still entangled

Suppose Alice goes to moon & measures her qubit, what happens?

50 % chance of measuring "$|0\rangle$" say

Now, joint state becomes $|00\rangle = |0\rangle \otimes |0\rangle$

Bob's qubit becomes $|0\rangle$ & this happens instanteously → Is this faster than light communication?

This is what Einstein called "spooky action at a distance"

One can make two arguments that there is no violations of physical rules here

① Alice doesn't really convey any information

When she measures, she gets a random bit which she doesn't apriori know

② There is a classical scenario which has the same outcome:

Suppose a coin is flipped & two coins with the same outcome are given to Alice & Bob each

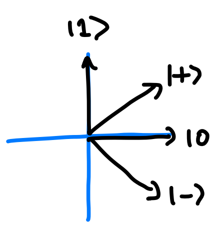
Alice doesn't look at her coin, until she gets to moon
When she looks at the coin, she knows Bob's outcome as well
but no physical rules are violated here

Such a theory is called a "Local Hidden Variable" theory

There are real states of the particles (as opposed to superposition) and we are only seeing probabilistic outcomes because we don't know the hidden variables

Einstein wanted the answer to be yes because of the following thought experiment by EPR:

(3) Suppose Alice measures her qubit in a different basis e.g. in $|\pm\rangle$ basis

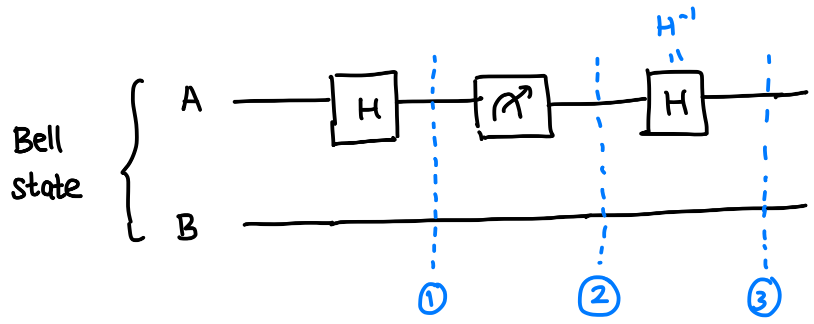


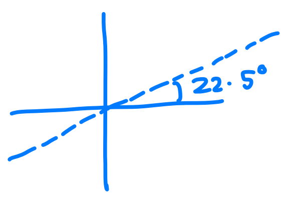
What happens to Bob's qubit?

Let's compute it in two different ways

$\boxed{\text{First}}$ Simulate $|\pm\rangle$ measurent with unitary + std. basis measurement

H = reflection at 22.5°





$H : |0\rangle \to |+\rangle$
$\quad |1\rangle \to |-\rangle$

$H^{-1} = H : |+\rangle \to |0\rangle$
$\quad |-\rangle \to |1\rangle$

State ① : $\frac{1}{\sqrt{2}}(H|0\rangle) \otimes |0\rangle + \frac{1}{\sqrt{2}}(H|1\rangle) \otimes |1\rangle$

$= \frac{1}{\sqrt{2}}|+\rangle \otimes |0\rangle + \frac{1}{\sqrt{2}}|-\rangle \otimes |1\rangle$

$= \frac{1}{2}|00\rangle + \frac{1}{2}|01\rangle + \frac{1}{2}|10\rangle - \frac{1}{2}|11\rangle$

State ② : A measures $\mathbb{P}[\text{ measures } 0] = \left(\frac{1}{2}\right)^2 + \left(\frac{1}{2}\right)^2 = \frac{1}{2}$

$\to$ state changes to $|0\rangle \otimes \left(\frac{1}{\sqrt{2}}|0\rangle + \frac{1}{\sqrt{2}}|1\rangle\right) = |0\rangle \otimes |+\rangle$

interprets as measuring "$|+\rangle$"

State ③ : Rotate back to get the correct state when we measure in the $|\pm\rangle$ basis

$\to$ state becomes $|+\rangle \otimes |+\rangle$ $\to$ final state

Bob's state is also $|+\rangle$

In the other case, with probability $1/2$, measures "1" $\to$ interprets as "$|-\rangle$"

final state is $|-\rangle \otimes |-\rangle$

This is similar to what happened if Alice measured in $\{|0\rangle, |1\rangle\}$ basis

Here, with 50% chance, she either gets a $|+\rangle$ or a $|-\rangle$
& Bob's state collapses to whatever Alice measures

$\boxed{\text{Second}}$ Let's do the above computation differently & directly try to measure in $|\pm\rangle$ basis

EPR pair : $\frac{1}{\sqrt{2}}|00\rangle + \frac{1}{\sqrt{2}}|11\rangle$

Let's express the first qubit in the $|\pm\rangle$ basis

$$|00\rangle = |0\rangle \otimes |0\rangle = \left(\frac{1}{\sqrt{2}}|+\rangle + \frac{1}{\sqrt{2}}|-\rangle\right) \otimes |0\rangle$$

$$|11\rangle = |1\rangle \otimes |1\rangle = \left(\frac{1}{\sqrt{2}}|+\rangle - \frac{1}{\sqrt{2}}|-\rangle\right) \otimes |1\rangle$$

$$\text{Final state} = \frac{1}{2}|+0\rangle + \frac{1}{2}|-0\rangle + \frac{1}{2}|+1\rangle - \frac{1}{2}|-1\rangle$$

$$= |+\rangle \otimes \left(\frac{1}{2}|0\rangle + \frac{1}{2}|1\rangle\right) + |-\rangle \otimes \left(\frac{1}{2}|0\rangle - \frac{1}{2}|1\rangle\right)$$

$$= \frac{1}{\sqrt{2}}|++\rangle + \frac{1}{\sqrt{2}}|--\rangle \quad !!!$$

EPR pair is an equal superposition of two different bases

If Alice measures in $|\pm\rangle$ basis :

$\mathbb{P}[\text{measures "}|+\rangle\text{"}] = \left(\frac{1}{\sqrt{2}}\right)^2 = \frac{1}{2}$ and similarly for the other case

and state collapses to $|+\rangle$

This is more spooky than before because Alice can maybe convey some information to Bob instanteously by deciding to measure either in $\{|0\rangle, |1\rangle\}$ or $\{|+\rangle, |-\rangle\}$ basis

Bob's state changes to something that Alice knows which is different depending on the basis

Has Alice managed to convey one bit of information to Bob via the following protocol :

Alice wants to send "0" to Bob : Measure in std. basis

$$\left.\begin{array}{l} 50\% \text{ chance : Bob's state becomes } |0\rangle \\ 50\% \text{ chance : ——————— } |1\rangle \end{array}\right\} \text{Mixed state } \rho_0$$

Alice wants to send "1" to Bob : Measure in $\{|\pm\rangle\}$ basis

$$\left.\begin{array}{l} 50\% \text{ chance : Bob's state becomes } |+\rangle \\ 50\% \text{ chance : ——————— } |-\rangle \end{array}\right\} \text{Mixed state } \rho_1$$

Bob does some local operation on his qubit to decode the message

Resolution : There is no local operation that Bob can do that distinguishes the two mixed states ρ_0 & ρ_1

NEXT TIME Quantum Mechanics is not a "Local Hidden Variable" theory
→ Entanglement is real