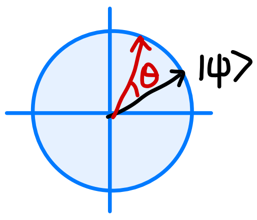
LECTURE 4   August 31, 2023

PART I   Fundamental Concepts in Quantum Information
- superposition
- Measurements
- "Quantum Operations"

This lecture
- Elitzur-Vaidman Puzzle (contd)
- Unitary Transformations & Multi-qubit systems

RECAP   How to transform state of a qubit?

Rotation

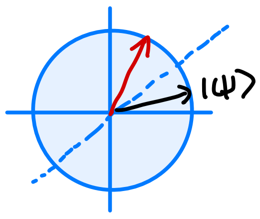


E.g. $R_\theta = \begin{bmatrix} \boxed{\begin{matrix}\cos\theta \\ \sin\theta\end{matrix}} & \boxed{\begin{matrix}-\sin\theta \\ \cos\theta\end{matrix}} \end{bmatrix}$

|0⟩ goes to |1⟩ goes to

$S = \begin{bmatrix} 1 & 0 \\ 0 & i \end{bmatrix}$

Reflection



E.g. Reflection around 45° line (NOT)

$|0\rangle \rightarrow |1\rangle$

$|1\rangle \rightarrow |0\rangle$

$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$

Also, called X gate

Reflection around 22.5° line (Hadamard H)

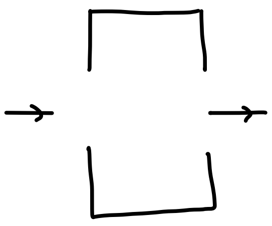
$|0\rangle \rightarrow |+\rangle$

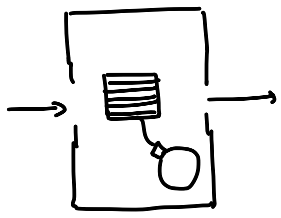
$|1\rangle \rightarrow |-\rangle$

$\begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & -1/\sqrt{2} \end{bmatrix}$

REVISIT   Elitzur-Vaidman Bomb Tester

Dud   OR   Bomb   fuse attached to "↔" filter





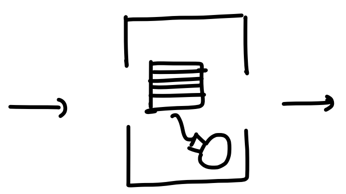
Nothing happens

Photon measured in $\{\overset{\leftrightarrow}{|0\rangle}, \overset{\updownarrow}{|1\rangle}\}$ basis & bomb explodes if |1⟩ is measured o/w photon comes out

Classically no chance of detecting

|+⟩ state gave us 25% chance

Let's try to give a better algorithm using the new operations introduced



- Start with $|0\rangle$
- Apply R_ε where $\varepsilon = \frac{\pi}{2n}$ for $n = 100000$
- Send into box
- If no explosion, repeat steps 2 and 3 n times
- Measure in standard basis

Case Dud : Qubit exits at angle ε

Case Bomb: $\mathbb{P}[\text{measure "}|0\rangle\text{"}] = (\cos\varepsilon)^2$ and then $|0\rangle$ exits

$\mathbb{P}[\text{measure "}|1\rangle\text{"}] = (\sin\varepsilon)^2 = \varepsilon^2$

If no explosion, photon comes out in state $|0\rangle$
Repeat steps 2 and 3 n times

Analyzing Full Algorithm

Case Dud : After n rotations, state of qubit is $|1\rangle$
since each rotation is $\frac{\pi}{2n}$

Case Bomb : Final state assuming no explosion is $|0\rangle$

$$\mathbb{P}[\text{explosion}] = n \cdot \varepsilon^2 = \frac{\pi^2}{4n} = \text{small}$$

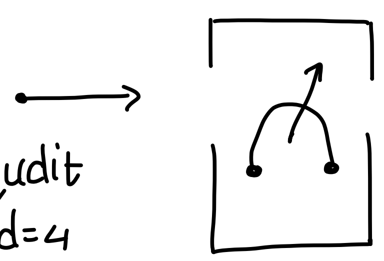
Measuring in standard basis : Dud $\longrightarrow |1\rangle$
Bomb $\longrightarrow |0\rangle$

Perfectly distinguish if there is no explosion

Rotation and Reflection operations are what are called unitary transformations!
We will talk about them more generally so let us first introduce a qudit.

d-Qudit A quantum system in superposition of d basic states $|1\rangle, |2\rangle, \ldots |d\rangle$

$$\begin{pmatrix}\alpha_1 \\ \vdots \\ \alpha_d\end{pmatrix} = |\psi\rangle = \alpha_1|1\rangle + \cdots + \alpha_d|d\rangle \quad \text{where } |\alpha_1|^2 + \cdots + |\alpha_d|^2 = 1$$



qudit
d=4

Measuring a qudit in the basis $\{|1\rangle, \ldots |d\rangle\}$

outcome is $|1\rangle$ with $|\alpha_1|^2$ & state collapses to $|1\rangle$
.... and so on

State of a qudit can also be changed by rotation/reflection in d-dimensions

QM law 3 A qudit state can be changed by any linear transformation that preserves length

These are called unitary transformations $\mathcal{U} \in \mathbb{C}^{d \times d}$

$$\mathcal{U} \text{ s.t. } \forall |\psi\rangle \quad \| \mathcal{U}|\psi\rangle \|^2 = \|\psi\|^2$$

$$\Leftrightarrow \quad (\mathcal{U}|\psi\rangle)^\dagger (U|\psi\rangle) = \langle\psi|\psi\rangle$$
$$\Leftrightarrow \quad \langle\psi| \underbrace{U^\dagger U}_{} |\psi\rangle = \langle\psi|\psi\rangle$$

This can only happen iff $U^\dagger U = I$

If $\mathcal{U} = \begin{pmatrix} | & & | \\ u_1 & \cdots & u_d \\ | & & | \end{pmatrix}$ then $\mathcal{U}^\dagger = \begin{pmatrix} — u_1^\dagger — \\ \vdots \\ — u_d^\dagger — \end{pmatrix}$

So, if $\mathcal{U}^\dagger \mathcal{U} = \begin{pmatrix} u_1^\dagger u_1 & u_1^\dagger u_2 & \cdots & u_1^\dagger u_d \\ u_2^\dagger u_1 & u_2^\dagger u_2 & \cdots & u_2^\dagger u_d \\ & \cdots & & \end{pmatrix} = \begin{pmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{pmatrix}$

then columns of $\mathcal{U}$ form an orthonormal basis

Another equivalent defn: $\mathcal{U}\mathcal{U}^\dagger = I \Leftrightarrow$ inverse of $U = U^\dagger$

This implies that if U is allowed, then so is U^{-1}

All unitary operations are reversible

Another equivalent defn: $\mathcal{U}$ preseves angles (or inner products)

$$(\mathcal{U}|\phi\rangle)^\dagger \mathcal{U}|\psi\rangle = \langle\phi|\mathcal{U}^\dagger \mathcal{U}|\psi\rangle = \langle\phi|\psi\rangle$$

E.g. (On qubits) $R_\theta = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$ → check that it is unitary

$$NOT = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

$$H = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$$

$$S = \begin{bmatrix} 1 & 0 \\ 0 & i \end{bmatrix} \qquad Z = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

E.g. (On qudits with $d=3$)

$$\begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} \alpha \\ \beta \\ \gamma \end{bmatrix} = \begin{bmatrix} \gamma \\ \alpha \\ \beta \end{bmatrix}$$

preserves length

Permutation Matrix

(Qudits with $d=4$)

$$\text{SWAP} = \begin{array}{c} \\ 00 \\ 01 \\ 10 \\ 11 \end{array} \begin{array}{c} \begin{array}{cccc} 00 & 01 & 10 & 11 \end{array} \\ \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \end{array}$$

$$H^{\otimes 2} = \frac{1}{2} \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \end{pmatrix}$$

Fun fact: Every unitary U has a square root

e.g. $\sqrt{R_\theta} = R_{\theta/2}$ and $\sqrt{NOT} = \frac{1}{2}\begin{pmatrix} 1+i & 1-i \\ 1-i & 1+i \end{pmatrix}$

Multi-qubit systems

Most common way of obtaining a qudit : 2 qubits

e.g. photon "0" = ↔ or "1" = ↕

state $\gamma_{00}|00\rangle + \gamma_{01}|01\rangle + \gamma_{10}|10\rangle + \gamma_{11}|11\rangle$

Say Alice has a qubit $|\psi\rangle$ and Bob has a qubit $|\phi\rangle \in \mathbb{C}^2$

Question 1 : What is the joint 4-d state?

Question 2: If Bob applies a unitary $U \in \mathbb{C}^{2\times 2}$ to his qubit, what is the new 4-d state?

Question 3 : If only Alice measures her qubit, what happens?

Lets try to answer question 1.

We can view two qubits as a joint 4-d system :

$$\gamma_{00}|00\rangle + \gamma_{01}|01\rangle + \gamma_{10}|10\rangle + \gamma_{11}|11\rangle$$

Alice's qubit   Bob's qubit

Say Alice's qubit $|\psi\rangle = \alpha_0|0\rangle + \alpha_1|1\rangle$

Bob's qubit $|\phi\rangle = \beta_0|0\rangle + \beta_1|1\rangle$

Analogous to the probability rules for flipping two coins,

Overall amplitude of $|00\rangle = \alpha_0\beta_0$

So, $\gamma_{00} = \alpha_0\beta_0$, $\gamma_{01} = \alpha_0\beta_1$, $\gamma_{10} = \alpha_1\beta_0$, $\gamma_{11} = \alpha_1\beta_1$

This better be a quantum state. let's check that

$$|\gamma_{00}|^2 + |\gamma_{01}|^2 + |\gamma_{10}|^2 + |\gamma_{11}|^2 = (|\alpha_0|^2 + |\alpha_1|^2)(|\beta_0|^2 + |\beta_1|^2) = 1 \cdot 1 = 1$$

More generally, what is the joint state of two qudits ?

$\boxed{\text{QM Law 4}}$ Alice d-qudit   Bob e-qudit

$$|\psi\rangle = \begin{array}{c} |1\rangle \\ \vdots \\ |d\rangle \end{array} \begin{bmatrix} \alpha_1 \\ \vdots \\ \alpha_d \end{bmatrix} \qquad |\phi\rangle = \begin{array}{c} |1\rangle \\ \vdots \\ |e\rangle \end{array} \begin{bmatrix} \beta_1 \\ \vdots \\ \beta_e \end{bmatrix}$$

Joint state is de-dimensional qudit

$$\begin{array}{c} |11\rangle \\ |12\rangle \\ \vdots \\ |1e\rangle \\ |21\rangle \\ \vdots \\ |de\rangle \end{array} \begin{bmatrix} \alpha_1\beta_1 \\ \alpha_1\beta_2 \\ \vdots \\ \alpha_1\beta_e \\ \alpha_2\beta_1 \\ \vdots \\ \alpha_d\beta_e \end{bmatrix} = \begin{bmatrix} \alpha_1 \begin{bmatrix} \beta_1 \\ \vdots \\ \beta_e \end{bmatrix} \\ \alpha_2 \begin{bmatrix} \beta_1 \\ \vdots \\ \beta_e \end{bmatrix} \\ \vdots \\ \alpha_d \begin{bmatrix} \beta_1 \\ \vdots \\ \beta_e \end{bmatrix} \end{bmatrix} = |\psi\rangle \otimes |\phi\rangle$$

This operation is called a tensor product.

More generally, tensor product of two matrices A and B:

$$A = \begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \\ a_{m1} & \cdots & a_{mn} \end{bmatrix} \qquad B = \begin{bmatrix} b_{11} & \cdots & b_{1q} \\ \vdots & & \\ b_{p1} & \cdots & b_{pq} \end{bmatrix}$$

$m \times n$ matrix $\qquad$ $p \times q$ matrix

$$A \otimes B = \begin{bmatrix} a_{11} B & a_{12} B & \cdots & a_{1n} B \\ & & & \\ & & & a_{mn} B \end{bmatrix}$$

Each block is a $p \times q$ matrix

$mp \times nq$ matrix

NEXT LECTURE

Multi-qubit system (contd)

Quantum Circuits & Entanglement