LECTURE 3 August 29^{th}, 2023

PART I Fundamental Concepts in Quantum Information

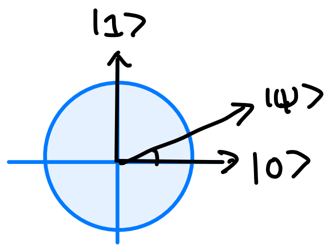
This Lecture Measurements in different basis & Global vs Relative Phase

Elitzur-Vaidman Bomb Tester

Unitary Transformations or Quantum Gates

$\boxed{\text{QM law 1}}$ Qubit can be in superposition of $|0\rangle$ & $|1\rangle$

$$\underbrace{|\psi\rangle}_{\begin{pmatrix}\alpha\\ \beta\end{pmatrix}} = \alpha\underbrace{|0\rangle}_{\begin{pmatrix}1\\ 0\end{pmatrix}} + \beta\underbrace{|1\rangle}_{\begin{pmatrix}0\\ 1\end{pmatrix}} \quad \text{where} \quad \underbrace{\langle\psi|}_{(\bar{\alpha}\ \bar{\beta})}\psi\rangle = |\alpha|^2 + |\beta|^2 = 1$$



$\boxed{\text{QM law 2}}$ (Standard) Measurement has two outcomes "$|0\rangle$" or "$|1\rangle$"

Born's rule

if $|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$

$\underbrace{\langle 0|\psi\rangle}$ = projection of $|0\rangle$ on $|\psi\rangle$ = cos (angle b/w $|0\rangle$ & $|\psi\rangle$)

measurement outcome is "$|0\rangle$" and similarly for "$|1\rangle$"

with prob. $|\alpha|^2$

& state "collapses" to $|0\rangle$

Exercise (In-class) (1) What is $\langle\psi|$ in terms of $\{|0\rangle, |1\rangle\}$ basis ?

(2) What is $|\psi\rangle\langle\psi|$?

Measurement wrt different basis $\{|b_0\rangle, |b_1\rangle\}$

$$|\psi\rangle = \alpha|b_0\rangle + \beta|b_1\rangle$$

Measurement outcome is "$|b_0\rangle$" and similarly for "$|b_1\rangle$"

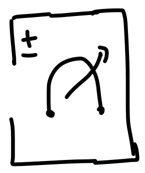
with prob. $|\alpha|^2$

state "collapses" to $|b_0\rangle$

Example

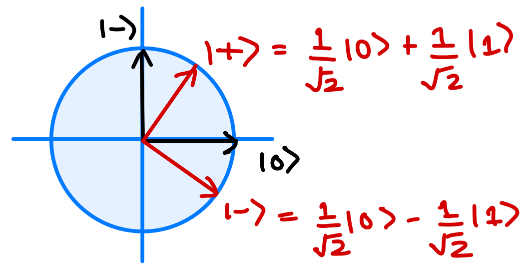
Outcome "$|+\rangle$" w.p. 1

$|+\rangle$ • $\longrightarrow$



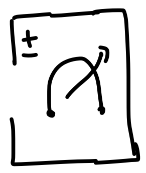
$\longrightarrow$ state "collapses" to $|+\rangle$

Measure in $\{|+\rangle, |-\rangle\}$



Can distinguish orthogonal states with probability 1

Outcome "$|+\rangle$" w.p. $\frac{1}{2}$ or "$|-\rangle$" w.p. $\frac{1}{2}$



$|0\rangle = \frac{1}{\sqrt{2}}|+\rangle + \frac{1}{\sqrt{2}}|-\rangle \longrightarrow$ [measurement] $\longrightarrow$ state collapses to $|\pm\rangle$

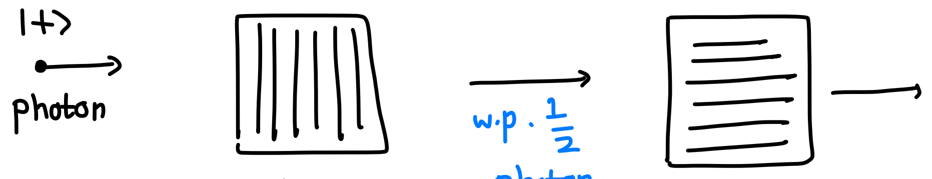
Measure in $\{|+\rangle, |-\rangle\}$ $\longrightarrow$ Hadamard basis

These example tell us the following:

If outcome in Hadamard basis is determined, then outcome in standard basis is uniform and vice versa

This is the "uncertainty principle"

Revisit Filter



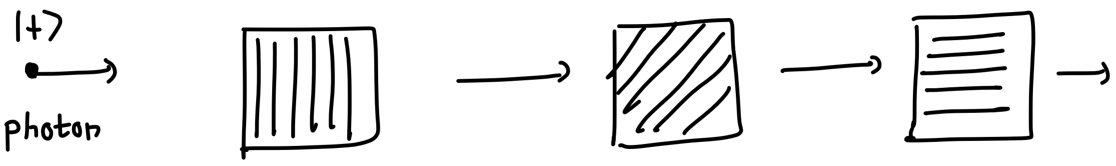
$\leftrightarrow = |0\rangle$

$\updownarrow = |1\rangle$

Measure in $\{|0\rangle, |1\rangle\}$ basis and block $|0\rangle$

w.p. $\frac{1}{2}$ photon in $|1\rangle$ comes out

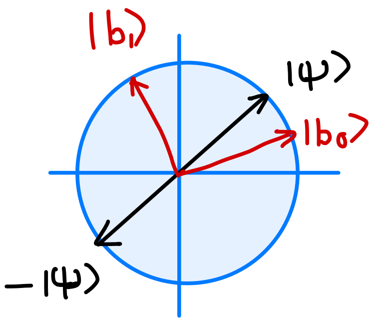
w.p. $|1\rangle$ no photon comes out



Measure in $\{|+\rangle, |-\rangle\}$ basis

What is the probability that the photon comes out?

Global Phase



Is there a difference between $|\psi\rangle$ and $-|\psi\rangle$?

No measurement can distinguish them

For any basis $\{|b_0\rangle, |b_1\rangle\}$ in which we measure

$$|\psi\rangle = \alpha|b_0\rangle + \beta|b_1\rangle$$
$$-|\psi\rangle = -\alpha|b_0\rangle - \beta|b_1\rangle$$

so prob. of outcomes is identical

In general, for any $\theta \in \mathbb{R}$

$$|\psi\rangle \text{ and } \underbrace{e^{i\theta}}_{\text{Global phase}}|\psi\rangle$$

cannot be distinguished

Relative Phase Are $|+\rangle = \frac{1}{\sqrt{2}}|0\rangle + \frac{1}{\sqrt{2}}|1\rangle$ and $|-\rangle = \frac{1}{\sqrt{2}}|0\rangle - \frac{1}{\sqrt{2}}|1\rangle$ the same ?

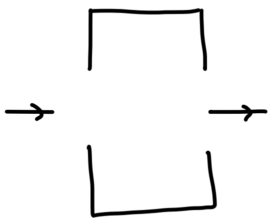
Relative phase

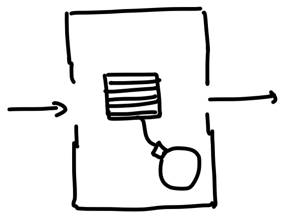
No! They can be distinguished w/prob 1 since they are orthogonal

Elitzur-Vaidman Bomb Tester

Suppose you are given a box which can be in one of two states

Dud OR Bomb fuse attached to "$\leftrightarrow$" filter





Nothing happens

Photon measured in $\{\overset{\leftrightarrow}{|0\rangle}, \overset{\updownarrow}{|1\rangle}\}$ basis & bomb explodes if $|1\rangle$ is measured o/w photon comes out

How can you test which box you are given ?

There is nothing you can do with "classical" strategies :

send in $|0\rangle \longrightarrow$ no information

send in $|1\rangle \longrightarrow$ explodes

Let's try "quantum strategies" :

- send in $|+\rangle$
- measure in $\{|+\rangle, |-\rangle\}$ basis

Case Dud : read $|+\rangle$ always

Case Bomb : $|+\rangle$ measured in $\{|0\rangle, |1\rangle\}$ basis

w.p. $1/2$ $|1\rangle \longrightarrow$ explosion

w.p. $1/2$ $|0\rangle \longrightarrow |+\rangle$ w.p. $1/2$

$|-\rangle$ w.p $1/2$ $\longrightarrow$ if you see this, you know it's a bomb

Summary : If there is a bomb, 50% chance of exploding

25% no explosion & detect bomb

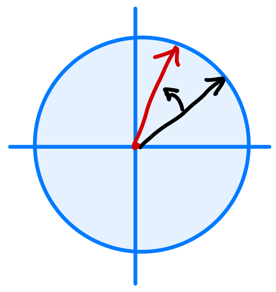
25% inconclusive

Later we will see how to improve it to 99% chance of detecting the bomb

Measurement gives us classical information and collapses the state

For quantum computing, we also need to be able to transform quantum states

Consider a qubit with real amplitudes



FACT For any θ, one can build a physical device that "rotates its state by θ"

E.g. by passing photon through a slab whose length depends on θ
or by shooting laser at an electron for time that depends on θ

The linear transformation that rotates by θ is given by the matrix

$$R_\theta = \begin{bmatrix} \boxed{\begin{matrix} \cos\theta \\ \sin\theta \end{matrix}} & \boxed{\begin{matrix} -\sin\theta \\ \cos\theta \end{matrix}} \end{bmatrix}$$

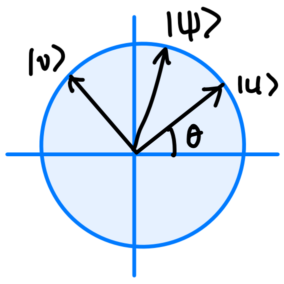
$\hookrightarrow$ where $|1\rangle$ goes

where $\begin{bmatrix} 1 \\ 0 \end{bmatrix} = |0\rangle$ goes

Same operation works for complex amplitudes also

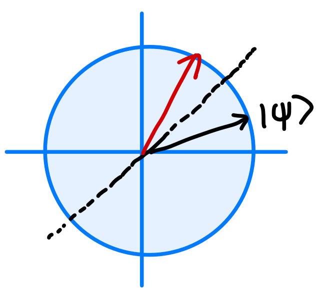
E.g. $\theta = 45^\circ$ $\quad R_{45^\circ} = \begin{bmatrix} 1/\sqrt{2} & -1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}$ $\quad |0\rangle \longrightarrow |+\rangle$, $\quad |1\rangle \longrightarrow -|-\rangle$

Can simulate measurement in any basis with Rotation operations and Standard measurements



- Pass $|\psi\rangle$ through $R_{-\theta}$
  - $|u\rangle \longrightarrow |0\rangle$
  - $|v\rangle \longrightarrow |1\rangle$
- Standard Measurement
  - "$|0\rangle$" means measured "$|u\rangle$"
  - "$|1\rangle$" means measured "$|v\rangle$"
- Apply R_θ to the collapsed state
  - $|0\rangle \longrightarrow |u\rangle$
  - $|1\rangle \longrightarrow |v\rangle$

FACT Can also build a physical device that implements a reflection



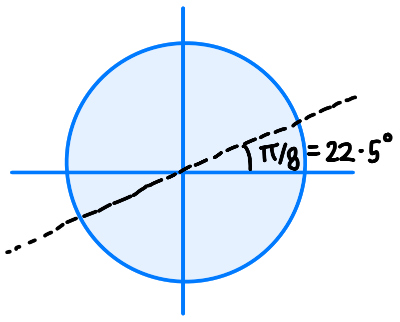
E.g. if state was $\begin{bmatrix} \alpha \\ \beta \end{bmatrix}$ & reflection thru 45°

state becomes $\begin{bmatrix} \beta \\ \alpha \end{bmatrix}$

The corresponding matrix is $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ NOT gate

$|0\rangle \rightarrow |1\rangle$
$|1\rangle \rightarrow |0\rangle$

E.g.



sends $|0\rangle \to |+\rangle$

$|1\rangle \to |-\rangle$

Matrix $\begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & -1/\sqrt{2} \end{bmatrix}$ Hadamard gate H

E.g. (with complex amplitudes) Phase shift operation

$$S = \begin{bmatrix} 1 & 0 \\ 0 & i \end{bmatrix} \qquad S \begin{bmatrix} \alpha \\ \beta \end{bmatrix} = \begin{bmatrix} \alpha \\ i\beta \end{bmatrix}$$

↓

valid qubit state

since $|\alpha|^2 + |i\beta|^2 = |\alpha|^2 + |\beta|^2 = 1$

$\boxed{\text{NEXT LECTURE}}$ An algorithm for Elitzur-Vaidman puzzle with small error

& more