LECTURE 12 September 28^{th}, 2023

PART II Fundamental Quantum Algorithms

Today Simon's Algorithm (wrapup)
Quantum Fourier Transform

RECAP Given black-box access to f that is L-periodic, determine $L \in \{0,1\}^n$

Black-box access : $\underbrace{|x\rangle}_{n}\underbrace{|y\rangle}_{m} - \boxed{U_f} - \underbrace{|x\rangle}_{n}\underbrace{|y \oplus f(x)\rangle}_{m=n-1}$

L-periodic : $f(x) = f(y)$ iff $x \oplus y = L$ $\Rightarrow$ pairs $(x, x+L)$ get a distinct color

$(1.4)^n$ classical vs $4n$ quantum

Key idea Use quantum subroutine to get random linear equations in bit representation of L

Quantum Subroutine

(i) Prepare the state $\frac{1}{\sqrt{2^n}} \sum_{x \in \{0,1\}^n} |x\rangle|f(x)\rangle$

(ii) Measure the COLOR c^*

$\Rightarrow$ state collapses to $\left(\frac{1}{\sqrt{2}}|x^*\rangle + \frac{1}{\sqrt{2}}|x^* \oplus L\rangle\right) \otimes |c^*\rangle$ where $f(x^*) = f(x^* + L) = c^*$

(iii) Applying $H^{\otimes n}$ to first n qubits gives $\frac{1}{\sqrt{2^{n-1}}} \sum_{s : s \cdot L = 0} (-1)^{x^* \cdot s} |s\rangle$ & MEASURE

Suppose we get $s^{(1)}, \ldots\ldots, s^{(T)} \in \{0,1\}^n$, then

$$\begin{bmatrix} — s^{(1)} — \\ — s^{(2)} — \\ \vdots \\ — s^{(T)} — \end{bmatrix} \cdot \begin{bmatrix} L_1 \\ \vdots \\ L_n \end{bmatrix} = 0$$

e.g.
$$\begin{aligned} L_1 \oplus L_2 \oplus L_3 \oplus L_4 &= 0 \\ L_2 \oplus L_3 &= 0 \\ L_1 \oplus L_2 &= 0 \\ L_3 \oplus L_4 &= 0 \end{aligned}$$

- At least 2 solutions 0, L and others
- If no equations, L could be one of 2^{n-1} possible colors
- Each new s if it is linearly independent, reduces # solns by half
- Exactly 2 solutions if $s^{(1)}, \ldots s^{(T)}$ contain $n-1$ linearly independent equations

Claim $\mathbb{P}[\text{first } n-1 \ s^{(1)}, \ldots s^{(n-1)} \text{ are linearly independent}] \geq \frac{1}{4}$

Start again if they are not

$\mathbb{E}[\#\text{ applications of } U_f \text{ until we succeed}] \leq 4n$

Proof of Claim Assume $s^{(1)}, \ldots, s^{(i)}$ are linearly independent
i.e. they span a subspace $\{\alpha_1 s^{(1)} + \alpha_2 s^{(2)} + \ldots + \alpha_i s^{(i)} \mid \alpha_1, \ldots \alpha_i \in \{0,1\}\}$
which has size 2^i

The next $s^{(i+1)}$ is independent if it is not in the span

$$\mathbb{P}[\underbrace{s^{(i+1)} \in \text{span}\{s^{(1)}, \ldots s^{(i)}\}}_{\text{bad event}}] = \frac{2^i}{2^{n-1}}$$

$$\mathbb{P}[\text{good: } s^{(i+1)} \notin \text{span}\{s^{(1)}, \ldots, s^{(i)}\}] = 1 - \frac{2^i}{2^{n-1}}$$

$$\mathbb{P}[\text{all } n-1 \text{ are linearly independent}] = \left(1-\frac{1}{2^{n-1}}\right)\left(1-\frac{2}{2^{n-1}}\right)\left(1-\frac{4}{2^{n-1}}\right)\cdots\left(1-\frac{2^{n-2}}{2^{n-1}}\right)$$

$\mathbb{P}[s^{(1)} \notin \text{span}\{0\}]$

$\mathbb{P}[s^{(n-1)} \notin \text{span}\{s^{(1)}, \ldots s^{(n-2)}\}]$

$$= \frac{1}{2} \cdot \frac{3}{4} \cdot \frac{7}{8} \cdots \left(1 - \frac{4}{2^{n-1}}\right)\left(1 - \frac{2}{2^{n-1}}\right)\left(1-\frac{1}{2^{n-1}}\right)$$

$$\geq \frac{1}{2}\left(1-\frac{1}{4}\right)\left(1-\frac{1}{8}\right)\left(1-\frac{1}{16}\right)\cdots$$

$$\geq \frac{1}{2}\left(1 - \frac{1}{4} - \frac{1}{8} - \frac{1}{16} - \cdots\right)$$

$$= \frac{1}{2} \cdot \frac{1}{2} = \frac{1}{4}$$ ∎

NEXT: Buildup to Shor's Factoring Algorithm

Given N, find p, q s.t. $p \cdot q = N$

Shor's algorithm uses a similar subroutine over integers mod N called ORDER FINDING

ORDER FINDING: $f(x) = a^x \bmod N$ where a is uniform random number coprime to N

find L (dividing N) s.t. $f(x) = f(x+L) = f(x+2L) = f(x+3L) \ldots$

$L \mid N$

Main differences with Simon's problem :

1. Arithmetic mod N where N is a large number
2. No promise that L divides N, so need some results from number theory to deal with it
3. We can build the black-box ourselves $\longrightarrow$ This is what makes it practical!

The algorithm for order finding is similar to Simon's algorithm but we need an analog of $H^{\otimes n}$ that works mod N

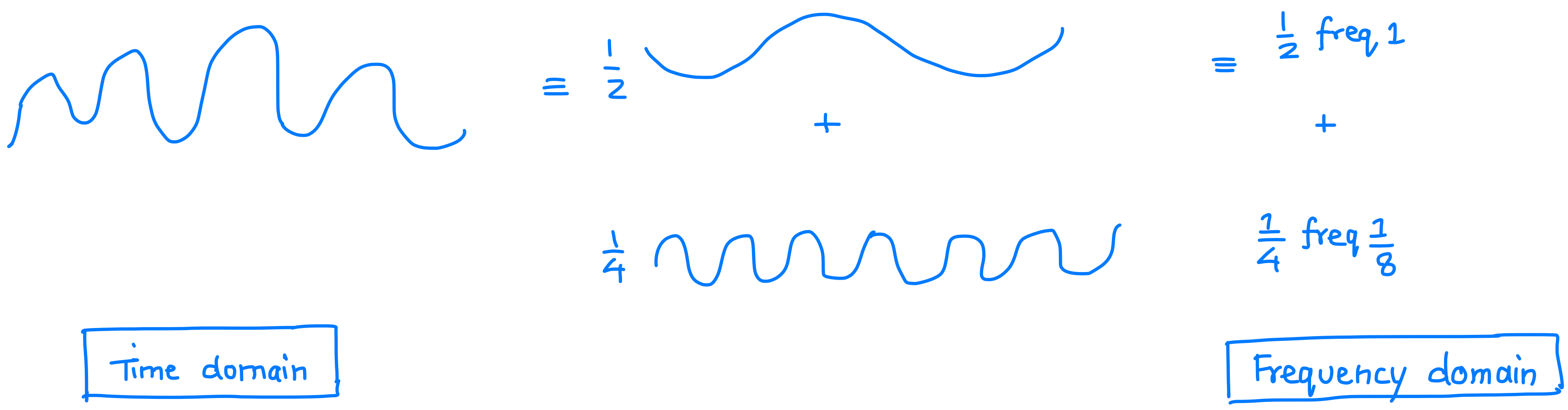
This is the Quantum Fourier Transform which we now introduce

Quantum Fourier Transform

Let us first talk about the classical discrete Fourier Transform

Useful in recovering periodic structure in data

E.g. continuous fourier transform allows



Discrete Fourier Transform (DFT)

Given $f : \mathbb{Z}_N \to \mathbb{C}$

$$\begin{array}{c} |0\rangle \\ \\ \\ |N-1\rangle \end{array} \begin{bmatrix} f(0) \\ \vdots \\ \vdots \\ f(N-1) \end{bmatrix} = \sum_{s=0}^{N-1} f(s)|s\rangle = \sum_{s=0}^{N-1} \hat{f}(s)|v_s\rangle$$

where $\{|v_0\rangle, \dots |v_{N-1}\rangle\}$ is a different basis called the $\mathbb{Z}_N$- Fourier basis and $\hat{f}(i)$ are the Fourier coefficients

$\downarrow$ "TIME" domain — Standard basis

$\downarrow$ "FREQ" domain — Fourier basis

(Inverse) DFT matrix $\quad DFT_N = \sum_{s=0}^{N-1} |v_s\rangle\langle s| = \begin{bmatrix} | & | & & | \\ |v_0\rangle & |v_1\rangle & \cdots & |v_{N-1}\rangle \\ | & | & & | \end{bmatrix}$

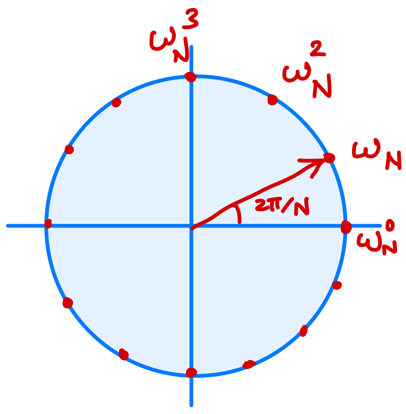
Unitary Matrix

$DFT_N^{-1} = DFT_N^{\dagger}$

E.g. N=2 $\quad DFT_2 = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} = H$

For general N, we need complex numbers

Let $\omega_N = e^{2\pi i/N}$ be the primitive N^{th}-root of unity



$\omega_N = e^{\frac{2\pi i}{N}} = \cos\left(\frac{2\pi}{N}\right) + i\sin\left(\frac{2\pi}{N}\right)$

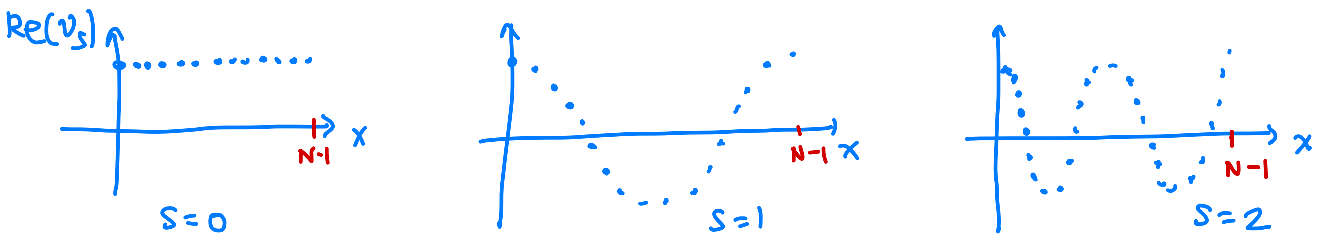
$\boxed{\text{FACT}}$ $\quad 1 + \omega_N + \omega_N^2 + \cdots + \omega_N^{N-1} = 0$

$= \frac{1-\omega_N^N}{1-\omega_N} = 0$

$$DFT_N = \frac{1}{\sqrt{N}} \begin{array}{c} \\ x \\ \\ \end{array} \!\!\begin{array}{c} s \\ \begin{bmatrix} & & \\ & \omega_N^{sx} & \\ & & \end{bmatrix} \end{array}$$

so, $\quad |v_s\rangle = \begin{bmatrix} \omega_N^0 \\ \omega_N^s \\ \omega_N^{2s} \\ \vdots \\ \omega_N^{(N-1)s} \end{bmatrix}$

Plotting Real parts of v_s the graph looks like a discrete cosine wave



E.g (N=4) $\quad DFT_4 = \frac{1}{\sqrt{4}} \begin{bmatrix} \omega_4^0 & \omega_4^0 & \omega_4^0 & \omega_4^0 \\ \omega_4^0 & \omega_4^1 & \omega_4^2 & \omega_4^3 \\ \omega_4^0 & \omega_4^2 & \omega_4^4 & \omega_4^6 \\ \omega_4^0 & \omega_4^3 & \omega_4^6 & \omega_4^9 \end{bmatrix} \rightarrow$ can express mod 4

since $\omega_4^4 = 1$

DFT_N^{-1} = Conjugate Transpose of DFT_4

= put negative signs in the exponent

One can compute discrete fourier transform of any vector in $\approx N\log N$ time classically

However, since DFT_N is a unitary matrix, one can applying it to a quantum state

NOTE The coefficients in standard and Fourier basis are encoded as amplitudes unlike the classical case where one can write the N coeffecients on a piece of paper

The advantage is that one can IMPLEMENT DFT_N for $N = 2^n$ with

$O(n^2)$ quantum gates (1 and 2 qubit gates)

$O(2^n \cdot n)$ time classically, so exponential savings but here we get a quantum state

Let's see how to do this by example, Say $N = 16$

We want to implement $|x\rangle \xrightarrow{DFT_{16}} \frac{1}{\sqrt{16}} \sum_{s=0}^{N-1} \omega_{16}^{sx} |s\rangle$ where $\omega_{16} = e^{\frac{2\pi i}{16}} := \omega$

$$DFT_{16}|x\rangle = \frac{1}{4}\left(|0000\rangle + \omega^{x}|0001\rangle + \omega^{2x}|0010\rangle + \omega^{3x}|0011\rangle + \dots + \omega^{15x}|1111\rangle\right)$$

Is this state entangled? No!

$$= \underbrace{\left(\frac{|0\rangle + \omega^{8x}|1\rangle}{\sqrt{2}}\right)}_{|s_3\rangle} \otimes \underbrace{\left(\frac{|0\rangle + \omega^{4x}|1\rangle}{\sqrt{2}}\right)}_{|s_2\rangle} \otimes \underbrace{\left(\frac{|0\rangle + \omega^{2x}|1\rangle}{\sqrt{2}}\right)}_{|s_1\rangle} \otimes \underbrace{\left(\frac{|0\rangle + \omega^{x}|1\rangle}{\sqrt{2}}\right)}_{|s_0\rangle}$$

Compare this to the following step in Simon's algorithm:

$$H^{\otimes n}|x\rangle = \underset{\uparrow \text{ if } x_2 = 1}{|+\rangle \otimes |-\rangle} \otimes \underset{\uparrow \text{ if } x_3 = 0}{|+\rangle} \otimes \dots$$

output qubit i depends only on input qubit x_i

For DFT, each output qubit depends on all n-input qubits

We will do the transform qubit-by-qubit

It will be very convenient to reverse the order

least significant bit $|x_0\rangle$ ———— $|s_3\rangle$ most sig.

$|x_1\rangle$ ———— $|s_2\rangle$

$|x_2\rangle$ ———— $|s_1\rangle$

most ——— $|x_3\rangle$ ———— $|s_0\rangle$ least sig

$x = x_3 x_2 x_1 x_0$

One can do $\frac{n}{2}$ SWAP gates to reverse the order at the end

To do the 0^{th} wire, we need to get $\frac{|0\rangle + \omega^{8x}|1\rangle}{\sqrt{2}}$ ← Seems like this depends on all 4 qubits of x

Notice, $\omega^8 = \omega_{16}^8 = (-1)$

So, $\omega^{8x} = (-1)^x$ and it only depends on whether x is even or odd, i.e. on x_0

So, we want $\frac{|0\rangle + (-1)^{x_0}|1\rangle}{\sqrt{2}} = H|x_0\rangle$

To do the 1^{st} wire, we need to get $\frac{|0\rangle + \omega^{4x}|1\rangle}{\sqrt{2}}$ ← Seems like this depends on all 4 qubits of x again

$\omega^4 = i$, so $\omega^{4x} = i^x$ ← only depends on $x \bmod 4$ i.e. x_0 and x_1

$$\omega^{4x} = \omega_{16}^{4(x_0 + 2x_1 + \cancel{4x_2} + \cancel{8x_3})}$$ since $16x_2, 32x_3 = 0$

$$= \omega^{4x_0} \cdot \omega^{8x_1} = (\omega^4)^{x_0} (-1)^{x_1}$$

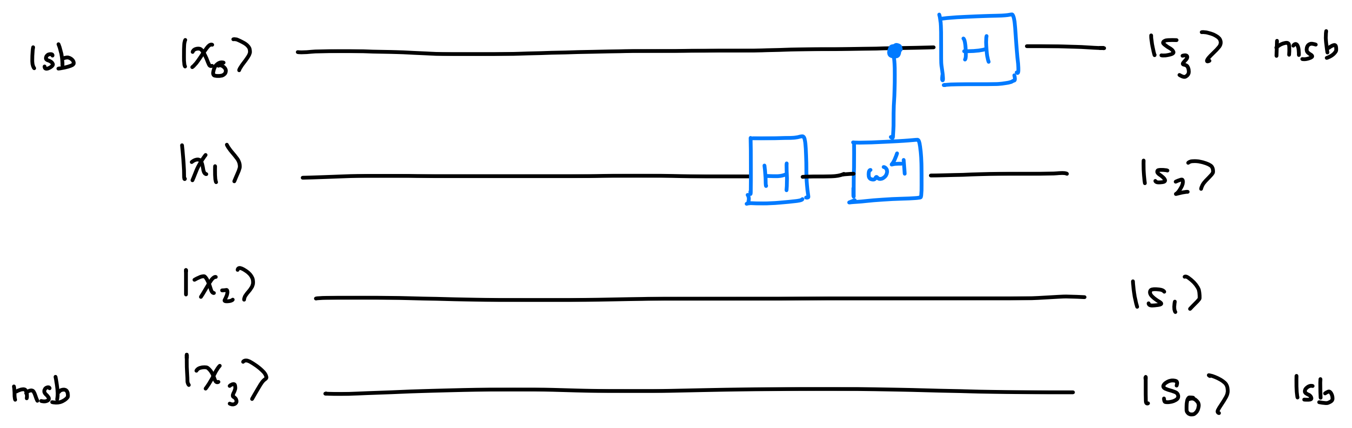
So, the $|1\rangle$ state should pick up phase (-1) if $x_1 = 1$ ← Hadamard
should also pick up phase ω^4 if $x_0 = 1$

"controlled - ω^4" gate, control qubit $= x_0$

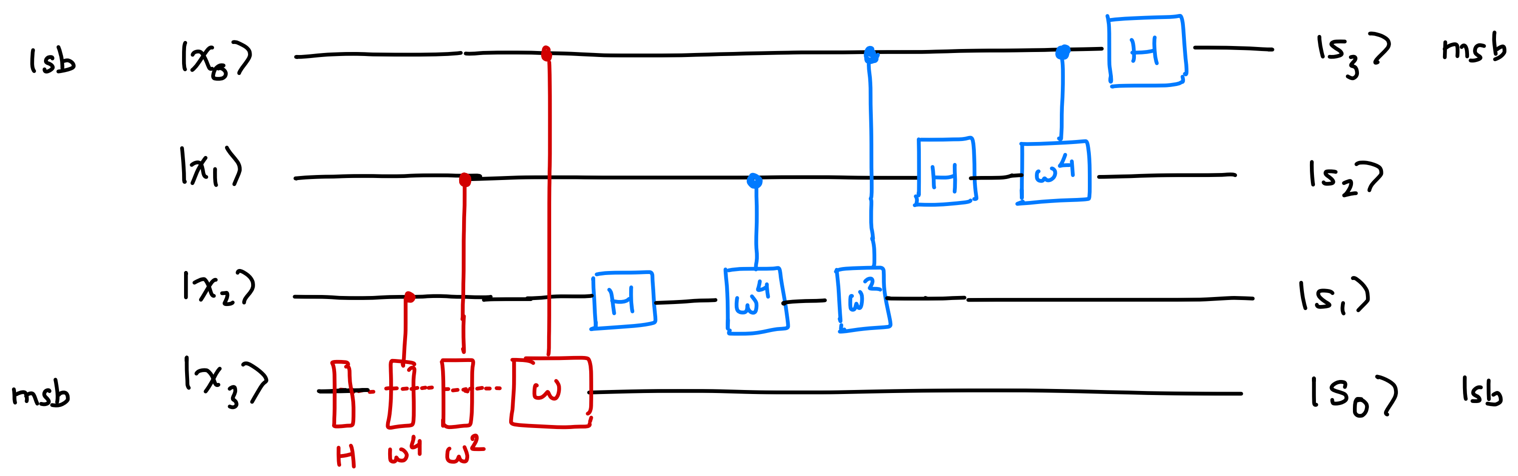
$|00\rangle \longrightarrow |00\rangle$ $\quad$ $|10\rangle \longrightarrow \omega^4|10\rangle$

$|01\rangle \longrightarrow |01\rangle$ $\quad$ $|11\rangle \longrightarrow \omega^4|11\rangle$

$$\begin{array}{c} 00 \\ 01 \\ 10 \\ 11 \end{array} \begin{bmatrix} 1 & & & \\ & 1 & & \\ & & \omega^4 & \\ & & & \omega^4 \end{bmatrix}$$



Rest is similar, in the end we have



Total gates : $1+2+3+4+\cdots+n = O(n^2)$

$\boxed{\text{Final Remarks}}$ For general n, say $n = 1000$ ω_{2^n} is the controlled 2^{1000}-th root of unity phase shift gate

We cannot build this accurately in practice

In general, not realistic for 2^k root of unity for $k \geq 30$

Luckily, it's not a problem!

$\boxed{\text{FACT}}$ Suppose we delete all gates where $k \geq \log\left(\frac{n}{\varepsilon}\right)$ E.g. $k=30$, $\varepsilon = 1\%$

Then, the resulting circuit

- "ε approximates" DFT_N $\longrightarrow$ success probability of Shor's algorithm only goes down by ε
- remaining gates can be built since they have large phases
- only $O\left(n \log\left(\frac{n}{\varepsilon}\right)\right)$ gates remain $\leftarrow$ Near linear size? Way more efficient!

$\boxed{\text{NEXT TIME}}$ Buildup to Shor's Algorithm : Order Finding