CS 476 Homework #11 Due 10:45am on 4/14

Note: Answers to the exercises listed below should be given in *typewritten form* (latex formatting preferred) by the deadline mentioned above. You should also email your answers and the relevant Maude code for Problem 2 to `cs476-staff@illinois.edu` (as explained below).

1. Call two equational theories (Σ, E) and (Σ, E'), *equivalent* (written $(\Sigma, E) \equiv (\Sigma, E')$), iff for each $(u = v) \in E$, $u =_{E'} v$ holds, and for each $(u' = v') \in E'$, $u' =_E v'$ holds. Intuitively, this means that (Σ, E) and (Σ, E') are "essentially" the same theory, but axiomatized with different sets of axioms E and E'. Prove:

   (a) If $(\Sigma, E) \equiv (\Sigma, E')$, then we have the equality of relations: $(\_ =_E \_) = (\_ =_{E'} \_)$ (that is, (Σ, E) and (Σ, E') prove the *same* equalities).

   (b) If $(\Sigma, E) \equiv (\Sigma, E')$, then $\mathcal{T}_{\Sigma/E} = \mathcal{T}_{\Sigma/E'}$ (that is, (Σ, E) and (Σ, E') have the *same* initial algebra).

   For Extra Credit. You can earn 10 more points on this problem (i.e., get a total of 20 points if you did everything perfectly) if you can prove that the equivalence:

   $$(\Sigma, E) \equiv (\Sigma, E') \quad \Leftrightarrow \quad \mathcal{T}_{\Sigma/E} = \mathcal{T}_{\Sigma/E'}$$

   is *false*, that is, only the implication $(\Rightarrow)$ (i.e., (b)) holds; but the implication $(\Leftarrow)$ fails in some cases.

2. Consider the theory of groups, which has an unsorted signature Σ with a constant 1, a unary function $(\_)^{-1}$ and a binary function $\_ \cdot \_$ and has the following equations G:

   (a) $x \cdot 1 = x$

   (b) $x \cdot (y \cdot z) = (x \cdot y) \cdot z$

   (c) $x \cdot (x)^{-1} = 1$

   Here is a good example of theory equivalence $(\Sigma, G) \equiv (\Sigma, G')$ and a very good reason for the alternative axiomatization (Σ, G') of the theory of groups: (Σ, G) is *not* confluent. Instead, you *already proved* in Homework # 10 that the following theory, (Σ, G'), written in Maude notation as follows (note the `fth` ... `endfth` keywords, since this theory has a "loose" semantics and *not* an initial algebra semantics) is locally confluent and terminating:

```
fth GROUP-CFL is
  sort Group .
  op 1 : -> Group .
  op i : Group -> Group .  *** inverse
  op _*_ : Group Group -> Group .

  vars x y z : Group .

  eq (x * y) * z = x * (y * z) .
  eq 1 * x = x .
  eq x * 1 = x .
  eq x * i(x) = 1 .
  eq i(x) * x = 1 .
  eq i(1) = 1 .
  eq i(i(x)) = x .
```

```
  eq i(x * y) = i(y) * i(x) .
  eq x * (i(x) * y) = y .
  eq i(x) * (x * y) = y .
endfth
```

In this exercise you are now asked to *prove* that these respective axiomatizations G and G' of group theory are *equivalent*, i.e.,

$$(\Sigma, G) \equiv (\Sigma, G').$$

Hints: (1). In proving that for each $(u = v) \in G$ $u' =_{G'} v'$ holds, you should be able to give a *mechanical proof* in Maude, since you know by Homework # 10 that (Σ, G') is a *decidable* theory. Therefore, you should include a screenshot of your mechanical proof and email your code for such a mechanical proof to `cs476-staff@illinois.edu`. (2). In proving that for each $(u' = v') \in G'$ $u' =_G v'$ holds, you can take advantage of the fact that in Homework #10 you proved using the axioms G the following theorems of Group Theory:

- $1 \cdot x = x$
- $(x)^{-1} \cdot x = 1$
- $(x \cdot y)^{-1} = y^{-1} \cdot x^{-1}$

.

Therefore, you can use the above theorems as *lemmas* in your proof that for each $(u' = v') \in G'$ $u' =_G v'$.