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Today

Bernoulli's equation

Betz's Law for maximum Rotor Efficiency

- We went through the derivation of the power present in the wind.
- We saw how this power is associated with the kenetic energy
- Now the question is: How much of this kenetic energy, or how much of this power in the wind stream can be converted to mechanical power?
- It turns out, we cannot extract all the power in the wind stream
- There is a limit, originally derived by a German physicist, Albert Betz (1919)

- We can get a qualitative feeling for what happens as the wind ~~speed~~ stream crosses the rotor: the wind speed decreases! → but never drops to zero

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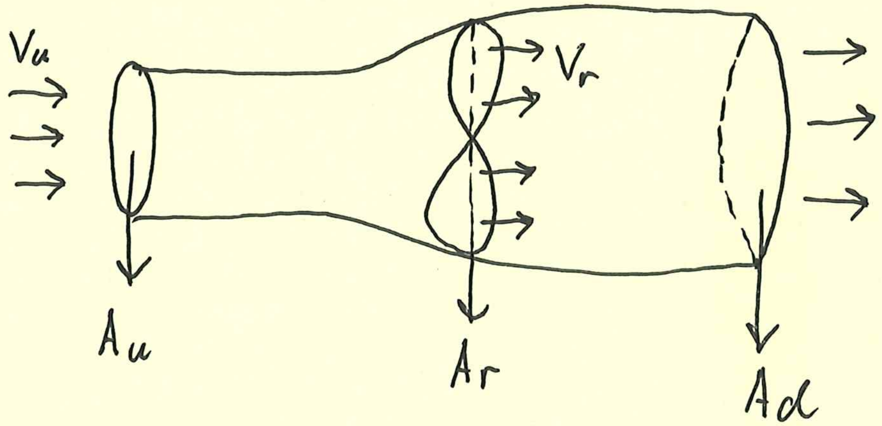


u: upstream
d: downstream
r: rotor
V = Velocity m/s
A = Area [m^2]

- Before the rotor: Kenetic energy $= \frac{1}{2} m V_u^2$
  - ↳ max energy extraction at the rotor would be if wind stops at the rotor $V_d = 0$
    - ↳ Not physically possible → the wind comming behind could not pass
  - However, if $V_d = V_u$ (no slowing) then energy extraction would be zero:
    - $\Rightarrow V_d < V_r < V_u$

Bernoulli's equation

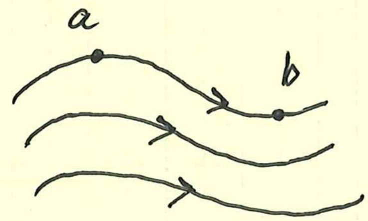
Bernoulli's principle

Bernoulli's Principle

- The total energy in a fluid is conserved if the flow obeys the following assumptions:

1) Points a and b lie on a streamline
2) Density, ρ, is constant (incompressible flow)
3) Steady flow (no turbulence)
4) No friction

$$\frac{1}{2}V^2 + gz + \frac{P}{\rho} = \text{const}$$



Streamlines - tangent to velocity vector of flow

$$\underbrace{P\overline{V}}_{\text{pressure Energy}} + \underbrace{\frac{1}{2}mV^2}_{\text{Kenetic Energy}} + \underbrace{mgh}_{\text{potential Energy}} = \text{constant}$$

In terms of mass that enters + exits the tube:

$$\frac{1}{2} m_a V_a^2 + g m_a h_a + P_a \underline{V}_a = \frac{1}{2} m_d V_d^2 + g m_d h_d + P_d \underline{V}_d$$

Volume of mass entering

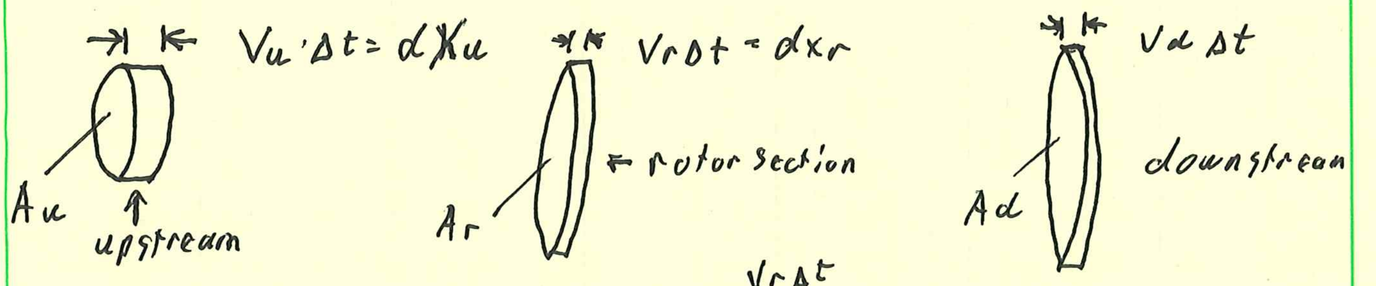
Volume of mass exiting

→ We want to calculate the power (or energy) that can be extracted from the wind as a function of the geometry, and upstream and downstream speeds

- We will use three conservation equations:
  - Mass
  - Energy
  - Momentum

~~We will assume: air is incompressible → ρ is constant~~

i) <u>conservation of mass</u>: the mass of an infinitesimal volume remains constant. Δt is a small increment of time



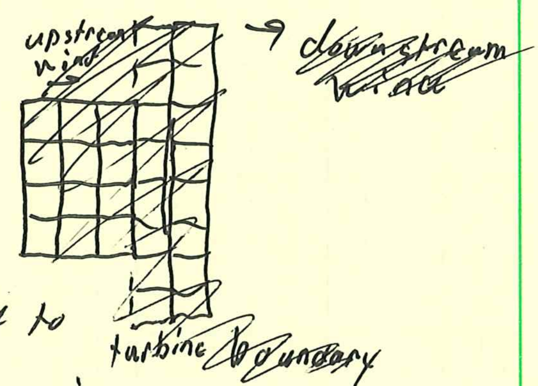
$$m_u = \rho_u \cdot A_u \cdot \underbrace{V_u \Delta t}_{dx_u} \qquad m_r = \rho_r \cdot A_r \cdot \underbrace{V_r \Delta t}_{dx_r} \qquad m_d = \rho_d\, A_d \cdot \underbrace{V_d \cdot \Delta t}_{dx_d}$$

conservation of mass

$$m_u = m_r = m_d \Rightarrow \cancel{\rho_u} \cdot A_u \cdot V_u \cdot \cancel{\Delta t} = \cancel{\rho_r} \cdot A_r \cdot V_r \cdot \cancel{\Delta t} = \cancel{\rho_d} \cdot A_d \cdot V_d \cdot \cancel{\Delta t}$$

$\rho_u = \rho_r = \rho_d = \rho$ (density is constant in tube)

$$\Rightarrow \boxed{A_u V_u = A_r V_r = A_d \cdot V_d}$$



ii) Conservation of Energy

Energy upstream is equal to energy downstream plus the energy delivered to the turbine rotor (ΔE_r) ~~as the wi~~.

$$E_u = E_d + \Delta E_r$$

$$E_u = \frac{1}{2} m_u \cdot V_u^2 + m_u \cdot g \cdot h_u + P_u \overbrace{A_u \cdot V_u \Delta t}^{\text{Volume}}$$

$$E_d = \frac{1}{2} m_d V_d^2 + m_d \cdot g \cdot h_d + P_d \cdot A_d \cdot V_d \cdot \Delta t$$

1. $P_u = P_d$ if we are far enough from turbine
2. By conservation of mass, then
   $$P_u \cdot A_u \cdot V_u \cdot \Delta t = P_d A_d V_d \cdot \Delta t$$
3. $Z_u = Z_d$ (upstream and downstream points are in the same horizontal plane
   $$m_u \cdot g \cdot Z_u = m_d \cdot g \cdot Z_d$$

thus: $\frac{1}{2} m_u V_u^2 = \frac{1}{2} m_d V_d^2 + \Delta E_r$

$$\Delta E_r = \frac{1}{2} m_u V_u^2 - \frac{1}{2} m_d V_d^2$$

$m_u = \rho V_u A_u \Delta t$, $m_d = \rho V_d \cdot A_d \cdot \Delta t$, $m_d = m_u$

$$\Delta E_r = \frac{1}{2} \rho A_r V_r [V_u^2 - V_d^2] \Delta t$$

The power extracted by the turbine is then:

$$\boxed{P_r = \lim_{\Delta t \to 0} \frac{\Delta E_r}{\Delta t} = \frac{1}{2} \rho A_r \cdot V_r [V_u^2 - V_d^2]}$$

We need to relate V_r with V_u and V_d.

It turns out that $V_r = \frac{V_u + V_d}{2}$.

We can derive this through Conservation of momentum.(iii) We won't though.

$$P_r = \frac{1}{2}\rho \cdot A_r \cdot \frac{V_u + V_d}{2}\left(V_u^2 - V_d^2\right)$$

Define ratio of downstream to upstream windspeed as:

$\lambda = \frac{V_d}{V_u}$ so: (lot's of Algebra)

$$P_r = \frac{1}{2}\rho \cdot A_r \cdot V_u^3 \underbrace{\left[\frac{1}{2}(1+\lambda)(1-\lambda^2)\right]}_{C_p}$$

$$\frac{dP_r}{d\lambda} = 0$$

$$\equiv \frac{dC_p}{d\lambda} = 0 \Rightarrow$$

$$\frac{d}{d\lambda}\left[\frac{1}{2}(1-\lambda^2+\lambda-\lambda^3)\right] =$$

$$\frac{1}{2}(1-2\lambda-\lambda^2) = 0 \Rightarrow \lambda \begin{cases} \to 1/3 \\ \to \cancel{0} \end{cases}$$

- does not make sense

$$\boxed{C_p^{max} = \frac{1}{2}(1+1/3)(1-(1/3)^2) = 0.593}$$

The max power that can be extracted is

$$P_r = 0.593 \cdot \frac{1}{2}\rho A_r \cdot V_u^3$$

$C_p^{max} = 0.593 =$ Betz Efficiency $= 16/27$

Most books, including ours, state that the Betz limit is the theoretical efficiency of the rotor. We need to be careful when reffering to this efficiency.

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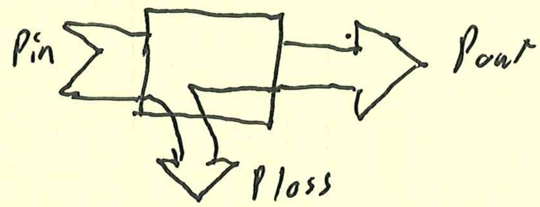
Efficiency can be defined as

$$\eta = \frac{P_{out}}{P_{in}}$$



Pout in our case is $P_r = 0.593 \cdot \frac{1}{2} A_r \rho V_u^3$

but what is Pin

two options:

$P_{in} = \frac{1}{2} \rho A_u \cdot V_u^3$ or $P_{in} \cancel{= \frac{1}{2}\rho} A_u$

power available in wind before being captured

$\Rightarrow$ not same as power in Betz expression:

$$\frac{1}{2} \rho A_r V_u^3$$

• If we use this definition of Pin, then given $\lambda = 1/3$

$V_d = \frac{1}{3} V_u$ and by conservation of mass

$$A_d \cdot V_d = A_r \cdot V_r \cdot A_u \cdot V_u$$

$\Rightarrow A_r = A_u \cdot \frac{V_u}{V_r}$ but $V_r = \frac{V_u + V_d}{2} = \frac{V_u + 1/3 V_u}{2} = \frac{2}{3} V_u$

$\Rightarrow A_r = \frac{3}{2} \cdot A_u$

$\Rightarrow \cancel{P_r = \frac{1}{2}\rho A_r V_u^3}$

$$\Rightarrow P_r = \frac{1}{2} \rho A_r V_u^3 (0.593) = \frac{1}{2} \rho A_u \cdot V_u^3 \cdot \underbrace{\frac{3}{2} (0.593)}_{0.889!}$$

$$\frac{P_{out}}{P_{in}} = 0.889$$

Another interpretation of Pin is the Power in the wind stream __before__ the turbine was placed.

In this case $V_r = V_u$
↑ physical location where we place turbine

Thus $P_{in} = \frac{1}{2}\rho A_u V_u^3 = \frac{1}{2}\rho A_r \cdot V_r^3 = \frac{1}{2}\rho \underbrace{A_r V_u^3}_{\text{term from Betz Law}}$

with this definition, $\eta = 0.593$

↳ ~~both~~ a either is valid, as long as you understand the assumptions that go with each

useful relations @ Betz efficiency

$V_r = \frac{2}{3} V_u \quad V_d = \frac{1}{3} V_u \quad V_d = \frac{V_r}{2}$

$A_r = \frac{3}{2} A_u \quad \underline{A_d = 3 A_u} \quad A_d = 2 A_r$

good HW problem